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MATHEMATICAL ANALYSIS OF THE LAMINATED ELASTOMERIC BEARING

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By

R. Clyde Herrick

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The purpose of the effort reported herein was to develop a mathematical model to assist in the design of laminated elastomeric bearings. The approach of the study was based upon the application of large-deformation elasticity and viscoelastic mathematical theories. However, the complexity of the large-deformation elasticity theory and the problems associated with computer solutions of equations of linear or classical elasticity for the particular bearing geometry and for materials that are almost incompressible were such that this analytical work was discontinued.

Although the effort reported herein did not result in attainment of the program objectives, the effort did result in the development of analytical approximations to the application of linear elastic theory and yielded results that are readily applicable and very close to exact solutions for bearings for light loads.

Future efforts by this command toward the development of design formulae for laminated elastomeric bearings will be limited to an experimental investigation approach.

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MATHEMATICAL ANALYSIS OF ~~THE~~ THE LAMINATED ELASTOMERIC BEARING

Final Report F-B2140-1

by

R. Clyde Herrick

Prepared by

The Franklin Institute
Laboratories for Research and Development
Philadelphia, Pennsylvania

for

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

ABSTRACT

The purpose of this study was the establishment of analytical design procedures for laminated elastomeric bearings. This was approached with the application of the linear mathematical theory of elasticity and later with nonlinear large-deformation elasticity theory.

The linear theory yielded analytical approximations that are close to exact solutions and which are easily applied and evaluated. This analysis of one typical lamination yields the distribution of stress and deformation in the elastomer between "rigid" metal lamina. However, the limits of the linear elasticity theory are exceeded for greater than small bearing loads, indicating the need for the application of the more comprehensive large-deformation elasticity theory.

The large-deformation theory was stated and the equilibrium equations were derived, but the solution of these equations was not carried out.

PREFACE

This report was prepared by The Franklin Institute Laboratories, Philadelphia, Pennsylvania, for the U. S. Army Aviation Materiel Laboratories under Contract DA 44-177-AMC-110(T).

The mathematical studies of the laminated elastomeric bearing presented herein, which began in September 1963 and were concluded in December 1964, represent the sole effort of The Franklin Institute, for which Mr. R. Clyde Herrick, Senior Research Engineer, Applied Mechanics Laboratory, was the principal investigator and author of this report. Acknowledgment is made to Mr. T. Y. Chu for the considerable contribution of the analytical approximations and computer solutions for the linear theory of elasticity shown in Appendixes A, B, and C, and for the brief presentation of pure torsion in large deformation elasticity shown in Appendix D. Acknowledgment is also made to Dr. Barry Wolf for the many formulations in large deformation elasticity leading to that presented in Appendix E and for his contributions to the work in linear theory done in conjunction with Mr. Chu. The contributions of Dr. Kishor D. Doshi and Zenons Zudans are also noted.

This constitutes the final report covering the first application of the mathematical theory of elasticity to the problem of design and laminated elastomeric bearings.

CONTENTS

ILLUSTRATIONS

SYMBOLS

Coordinate System

r, θ, z = cylindrical coordinates

u, v, w = displacement in the radial, tangential and axial directions, respectively

r_i = inside radius

r_o = outside radius

h = one-half the thickness of elastomer per lamination

$\pm w_o$ = axial displacement at $z = \pm h$

Stress

σ_{ij} = general component of stress

σ_{rr} = normal stress in r direction

σ_{rz} = shear stress parallel to rz plane on a surface whose surface normal is a radial line

Strain

ϵ_{ij} = general component of strain

ϵ_{rr} = normal strain, denoted by repeated subscript of coordinate system symbols

ϵ_{rz} = shear strain

ϵ_{kk} = dilation, $\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}$, denoted by a repeated subscript with other than the coordinate symbols, r, θ, z

Material Constants

G = shear modulus

ν = Poisson's ratio

k = bulk modulus

λ, μ = Lamé material constants

<u>Nomenclature Unique to Appendixes D and E</u>

τ^{ij} = tensor components of stress

σ_{ij} = physical components of stress

$G_{ij}, G^{ij}, g_{ij}, g^{ij}$ = metric tensors

(r, θ', z') = undeformed coordinates

(ρ, θ, z) = deformed coordinates

Φ, Ψ = constants from constitutive relationship

$\Gamma_i^{\ jm}$ = Christoffel symbols

P = pressure

SUMMARY

The purpose of this work was to establish a design procedure for laminated elastomeric bearings based upon the application of large deformation elasticity and viscoelastic mathematical theories. However, the complexity of the large deformation elasticity theory and certain problems associated with computer solutions of the equations of linear or classical elasticity for the particular bearing geometry and for materials that are almost incompressible were such that this work could not be completed.

In the course of this effort, analytical approximations to the application of linear elasticity theory were developed that not only yield results that are very close to exact solutions but are easily applied and evaluated. However, these approximations are for linear elasticity theory, the limits of which are exceeded for greater than very light bearing loads. While not quantitatively applicable for bearings under full loads, they do give much information concerning elastomer behavior under light loads.

The equilibrium equations for large deformation elasticity theory in cylindrical coordinates were stated for combined axial compression and torsion, but were fully derived for the case of compression only.

CONCLUSIONS

While the objectives of the study were not reached, much useful knowledge about the behavior of the elastomer in each lamination under load was generated. The approximate theories, (a) for incompressible materials and (b) for almost incompressible materials, are useful to indicate the relative effect upon bearing performance of bulk modulus (compressibility), ratio of inside diameter to outside diameter, and width-to-thickness ratio. More important, much was learned with respect to techniques for obtaining a useful solution of the nonlinear large-deformation equations that is necessary to predict adequately the distribution of displacement and stress within the bearing, especially along the bonded surface under realistic loading conditions. The original beliefs that large deformations are present within the elastomer even for very light compressive loads were confirmed. But, although much is known as to the behavior of the elastomer under load from the application of linear elasticity theory, this must be conditioned with the statement that this is qualitative and not quantitative for full loads.

INTRODUCTION

During the experimental development of the laminated elastomeric bearing for low-temperature applications (Technical Documentary Report No. ASD-TDR-63-769, prepared for Wright-Patterson Air Force Base, Ohio, November 1963), the need for a mathematical analysis of the bearing which would form the basis of a design method was recognized. Of primary interest were the magnitude and distribution of shear and normal stresses between the elastomer and the metal lamina. This was not only to enable a designer to specify elastomers with adequate strength and adhesive strength (at bond to metal lamina), but to determine the minimum thickness of metal lamina that could be used. Only by a mathematical analysis could the influence of compressive load, shear load (torsion), ratio of outside diameter to inside diameter, width-to-thickness ratio, and shear and bulk moduli of the elastomer be studied.

It was also anticipated that the deformations and strains within the elastomer were large and that the more general mathematical theories of elasticity would be required for an analysis, especially if the interaction of compressive load and bearing torsion observed in the experimental development were to be studied. Linear or classical theory is made linear by the assumptions that both deformation and strain are infinitesimal, thereby removing the ability to study the interaction of axial compression and torsion.

It was anticipated that in this study the state of stress and strain in the bearing could be investigated sufficiently with the use of large-deformation elasticity and viscoelasticity theories and that a computer program could be devised for the design of optimized bearing laminations for a given application and loading condition.

REVIEW OF EFFORT

When the present program was begun, the intention was to go directly to large-deformation elasticity because it appeared that sufficient theory was at hand. The intention was to use the linear or classical theory only for guidance and to establish certain "known points" in order to check the large-deformation work.

Soon after the large-deformation formulation was begun, it was discovered that a simplification could be made if it could be said that plane surfaces in the elastomer parallel to the bonded boundaries remained flat planes after deformation. Some preliminary work with the linear theory of elasticity indicated that these surfaces did not remain flat planes at the free edge where interest in stress and deformation is greater, and so the formulation of the large-deformation problem was made more general. By this time it was apparent that the equations for large deformations would be made up of many terms and that solutions would be time consuming.

At this time during the program, it was believed that an investigation should be made of the effects of small amounts of compressibility of the elastomer upon the displacement pattern and consequently upon stress magnitude and distribution. This was needed because the only constitutive relations (stress-strain relations) that were immediately available were the "Mooney Relations" (Reference 5) that were formulated for incompressible rubber-like materials. No relationships were at hand for compressible materials, because it seems that most rubber-like materials are used under such loading conditions that the hydrostatic component of total stress is relatively small. Hence, volume compression is small and may be neglected.

An investigation into the effect of small amounts of compressibility, using classical elasticity theory, was then begun. First, the literature was searched for solutions to similar elasticity problems (the problems of compression of a cylinder with ends fixed for no radial displacement). One similar problem (Reference 6) was discovered, and although the boundary condition was that of no radial displacement at the outside edge only, instead of along the whole bonded surface, it was believed that this solution from the literature would be useful. However, although it was an analytical solution, the effort necessary to extract information, even by a computer, for various conditions proved to be considerable. It was

much beyond the effort considered necessary to make an independent solution of the equilibrium equations of elasticity for the real boundary conditions (a bonded surface) by means of a digital computer. Therefore, this latter approach was started.

The first choice was to solve the equations of a compressible material. Appendix A shows the formulation of the equilibrium equations and the complete computer program (in FORTRAN) for the iteration procedure to yield deformations. In the theory of elasticity for homogeneous and isotropic materials, two constants are necessary to describe the material behavior. Of the sets of two constants that can be used, G, the shear modulus, and ν, Poisson's ratio, were chosen because the value of Poisson's ratio reflects the relative amount of compressibility: $\nu = 0.5$ for incompressible materials, and $\nu < 0.5$ for compressible materials. Since the equilibrium equations in terms of displacements contain the coefficient $1/1-2\nu$, which approaches infinity as ν approaches 0.5, it was decided for the first solution that a moderately compressible material, say, $\nu = 0.35$, should be used. The value of ν could then be increased to as close to 0.5 as possible in later solutions, after check-out and proof of the iteration procedure. Except for some initial trouble in programming finite-difference methods for the geometry of a very wide, but thin, annulus, the iteration procedure worked well. However, on subsequent trials, the iteration method would not converge to a reasonable solution of Poisson's ratios greater than $\nu = 0.4$. After this was worked with for a period of time, effort was stopped on this computer solution for the compressible elasticity theory, and all furtner investigation of compressible materials was accomplished with the use of an approximate solution described below.

The next effort was the formulation of the equations for incompressible elasticity theory and the computer solution thereof. This required a completely new formulation because the coefficient, $1/1-2\nu$, is infinite for $\nu = 0.5$. Consequently, the new formulation introduced a pressure term (Reference 1, page 79, problem 4) in the displacement relations instead of a volume compression term. Appendix B shows the derivation of these partial differential equations and the FORTRAN program for the solution by iteration.

The iteration scheme proved to be erratic. While effort was being made to improve the convergence of the method, an effort was made concurrently toward an approximate solution, also shown in Appendix B. It was not until some results of the approximate theory were used as the starting values for iteration that convergence seemed probable. By this time considerable effort had been expended upon this solution, because the approach was contined inasmuch as each innovation introduced seemed

to improve the results. The last computation made, of which partial computer output is included in Appendix B, appears to have been converging as planned, although the convergence was very slow.

In the meantime, approximate solutions, both analytical and computer, to the compressible cases of linear elasticity were investigated. It was recognized that the exact solution of the partial differential equations of equilibrium was not of major interest in this study but was only to guide the formulation and solution of the large-deformation case. The search for approximate solutions was fruitful. An analytical solution based upon the linear theory of elasticity was derived, not only for the incompressible case but for the compressible case as well; that is, flexible enough to accept Poisson's ratios in the neighborhood of almost negligible compressibility, $\nu = 0.495$ and $\nu = 0.499$. These values represent the order of magnitude of Poisson's ratios for elastomers of interest in the laminated elastomeric bearing. These approximations for incompressible and compressible theory are shown in Appendixes B and C, respectively.

Appendix C, for the approximate compressible theory, includes an attempt made to solve equations (C-17a) and (C-17b) by computer, using the Rung-Kutta-Gill method. The flow chart and computer program (FORTRAN) are shown at the end of Appendix C. The computer solution did not work, however, and the reason was discovered from the analytical solution: the solution consisted of e^{+x} and e^{-x}, with x assuming very large values. Inasmuch as this approximate solution for compressible materials, as shown by the analytical solution in Appendix C, was obtained by applying the variational theorems of linear elasticity, it satisfies equilibrium approximately, and it satisfies the stress boundary conditions on the average, that is, across the thickness.

Appendix D shows the derivation of equations, based upon large-deformation elastic theory, for the case of torsion only. This was done mainly for investigation of the large-deformation problem of combined compression and torsion. Note that it is infinitely less complex than the case of large-deformation compression, as shown in Appendix E.

The real stumbling block in this study is the set of large-deformation equations for axial loading (Appendix E). These involve constitutive relations (stress-strain relationships) that are, as yet, not known for the compressible case although they are generally known for the incompressible case. The real problem is the size and complexity of the highly nonlinear equations and the consequent lack of assurance that a solution is the right solution. This is the basis of the desire to establish solutions for light loads using classical elasticity, the establishment of a check point.

RESULTS OF LINEAR THEORY

While the computer solutions did finally yield satisfactory results for the incompressible case and for the compressible case where Poisson's ratio was 0. 4 or less, these displacements or deformations were not translated into stress. By the time that it could have been done with reliable computer solutions, the analytical approximations were available. These yield a solution that is closer to the exact solution of the equations of linear elasticity than the application of linear elasticity theory is to the real bearing. Perhaps it should be mentioned here that the linear theory of elasticity is believed to yield a very close approximation to elastomer stress and deformation for light loads, but only for light loads, as will be shown presently.

SOLUTIONS FOR INCOMPRESSIBLE ELASTOMERS

Results of the computer solution (Appendix A) for the compressible case are not shown here because, for practical elastomers, the ratio of bulk modulus to shear modulus is so very large that the behavior is more like that of an incompressible material. While the computer solution in Appendix A worked, it only worked for materials with a Poisson's ratio up to 0. 40. This is too low for elastomers of immediate interest.

Since meaninful results were obtained from the computer solution for an incompressible material and since a useful approximation was also derived, the results of the work shown in Appendix B are summarized as follows:

1. By examining successive iterations, it was found that the iterative procedure as programmed in Appendix B converges. The convergence is much improved if the iterative procedure is started with the results of the approximate solution, but it is still slow. This is evidenced by the fact that the computer output shown in Appendix B is from the 600th iteration, and certain inconsistencies are still present at the inner and outer boundaries. However, this disturbance is seen to exist at the free boundaries and propagates only about one-half a thickness (2h) into the bearing from either boundary; therefore, this was accepted as a good solution.

2. The influences of the boundaries $r = r_i$ and $r = r_c$ are manifested only near those boundaries. Away from the bound ies by, say, 5 thicknesses, the approximate solution and the computer coincide. Thus, the approximate solution can be used to predict all regions of the bearing except near the boundaries $r = r_i$ and $r = r_o$. These can be investigated by computer solutions using much funer gridworks.

3. Within the limits of the linear theory of elasticity, which has been grossly exceeded in the computer example, the analytical approximation, as shown in equations (B-7) through (B-23), is a very accurate method.

Figures 1 through 4 present data gained from the analytical approximations. Figure 1 shows for incompressible materials the relationship between average pressure, width-to-thickness ratio, and ratio of axial displacement to elastomer thickness. Thus, for

$$G = 80 \text{ psi (shear modulus)}$$

$$\frac{r_o - r_i}{2h} = 250 \text{ (width-to-thickness ratio)}$$

$$\frac{w_o}{h} = 0.002 \text{ (ratio of axial deformation to elastomer thickness),}$$

then

$$P_{ave} = 125G = 10,000 \text{ psi.}$$

However, from Figure 2, it is estimated that the associated maximum shear strain is $\epsilon_{rz} = \partial u/\partial z = 1.63$. This strain is far beyond the limits of the linear theory of elasticity because the linear theory is based on the assumption that $\partial u/\partial z$ is very small, as compared to 1.0.

From Figure 3, a measure of what is meant by "light loads" associated with linear elasticity theory can be obtained. Although $\partial u/\partial z = 0.2$ is still much beyond the values acceptable within the linear theory of elasticity, it is not sufficiently large to change the order of magnitude of the results. Assume, then, that $\partial u/\partial z = 0.2$ is accepted. From Figure 3, for a width-to-thickness ratio of 250, $P_{ave} = 15.5$ G. Thus, if G = 80 psi, as for the silicone rubber used in previous programs, then average pressure is 1240 psi. A bearing of this geometry and these materials was tested to an average pressure in excess of 40,000 psi. The rubber did not extrude or come unbonded, but the steel lamina broke. Thus, it is

known that elastomers with this geometry are capable of extreme pressures and extreme shear strains.

For strains that are not excessively large, say, $\epsilon_{rz} = 0.2$, a relationship between average pressure and the maximum shear stress as derived from the incompressible elastic approximation can be shown.

When

$$\frac{r_o - r_i}{2h} = 250 \text{ and } \frac{W_o}{h} = 0.00025,$$

then

$$\epsilon_{rz} = \frac{\partial u}{\partial z} = 0.2$$

and

$$\frac{P_{ave}}{G} = 15.5 .$$

Shear stress is

$$\sigma_{rz_{max}} = 2G\, \epsilon_{rz} = 0.4\, G ;$$

then

$$\frac{\sigma_{rz_{max}}}{P_{ave}} = \frac{0.4}{15.5}$$

and

$$\sigma_{rz_{max}} = 0.0258\, P_{ave}.$$

This provides some measure of the relationship between shear stress and average pressure, although the limits of linear elasticity have already been exceeded at this strain level.

APPROXIMATE SOLUTION FOR COMPRESSIBLE ELASTOMERS

The approximation shown in Appendix C has also yielded convenient expressions for the investigation of stress and displacement within the elastomer of a lamination. These equations are summarized as follows:

for $\sigma_{z_{average}}$, use equation (C-47);

for $\sigma_{z_{max}}$, use equation (C-48);

for ratio $\frac{\sigma_{z_{max}}}{\sigma_{z_{avg}}}$, use equation (C-49);

and

for ratio $\frac{\sigma_{rz_{max}}}{\sigma_{z_{avg}}}$, use equation (C-50).

Figure 4 shows a comparison of data for compressible and incompressible theory. The relative position of the curves indicates the influence of various amounts of compressibility, although the exact placement of the curves is not necessarily accurate because they have been calculated from linear theory where the strains exceed those allowable.

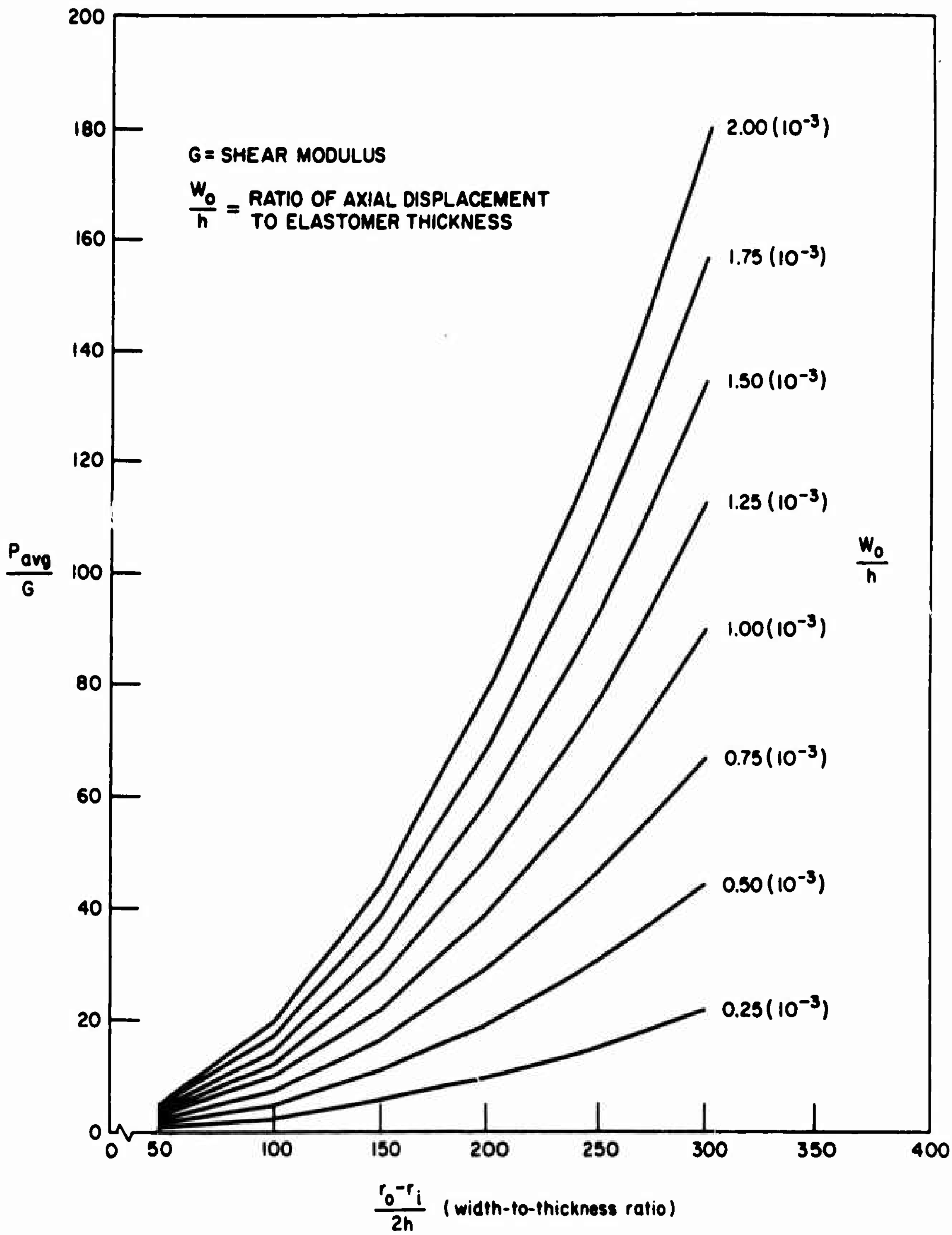


Figure 1. Average Pressure Versus Axial Compression (Incompressible Elasticity Theory).

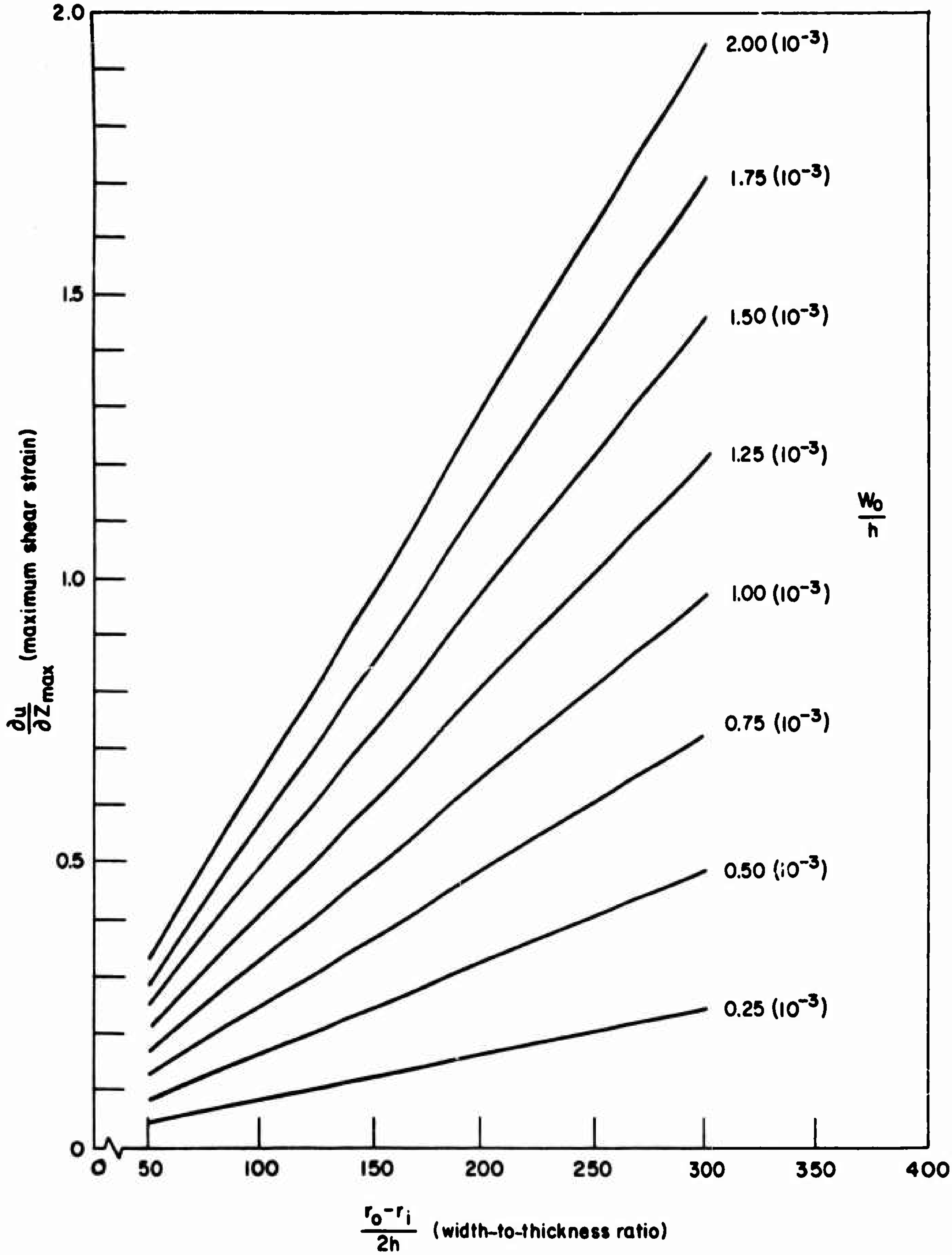


Figure 2. Shear Strain Versus Axial Compression (Incompressible Elasticity Theory).

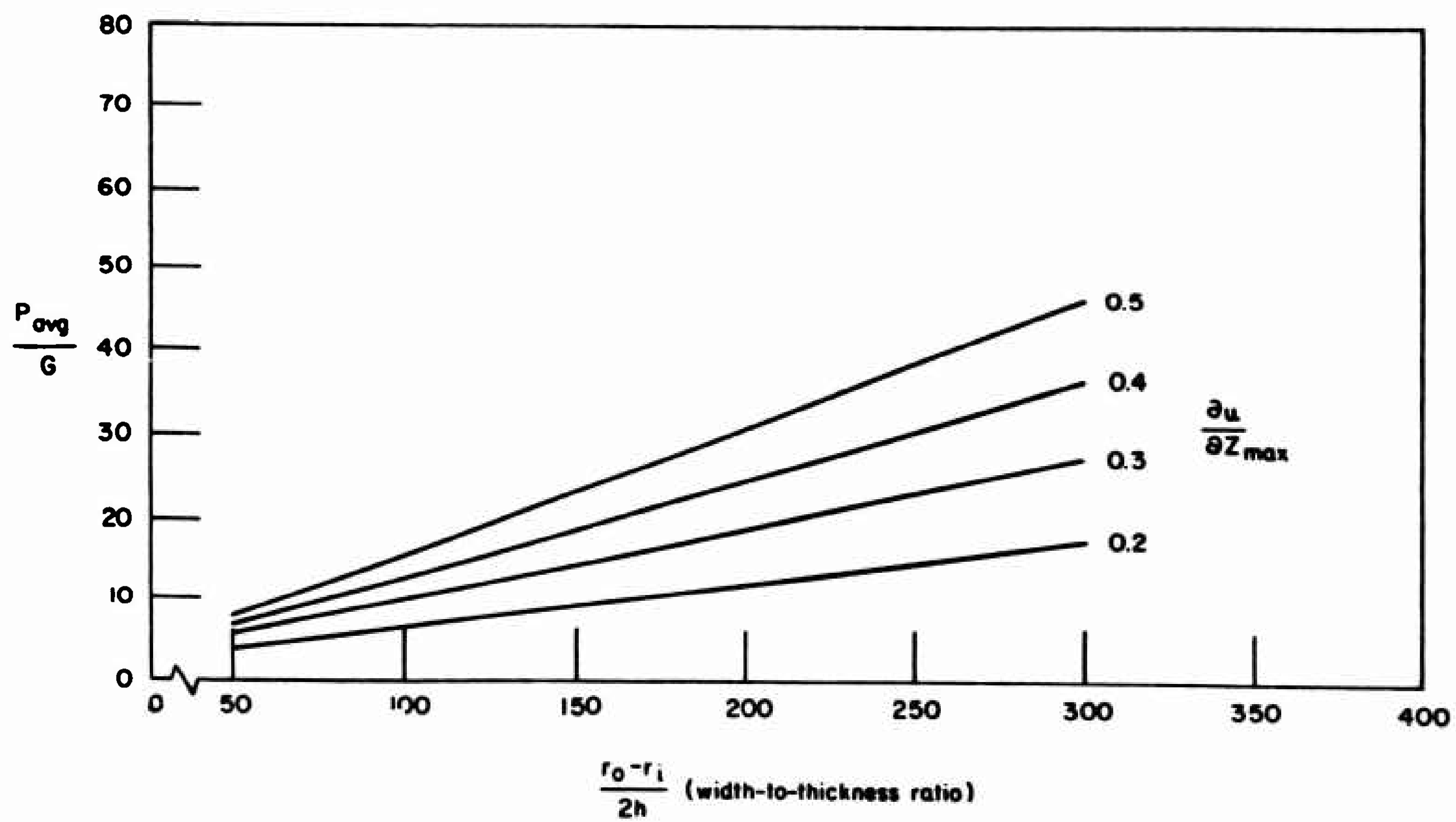
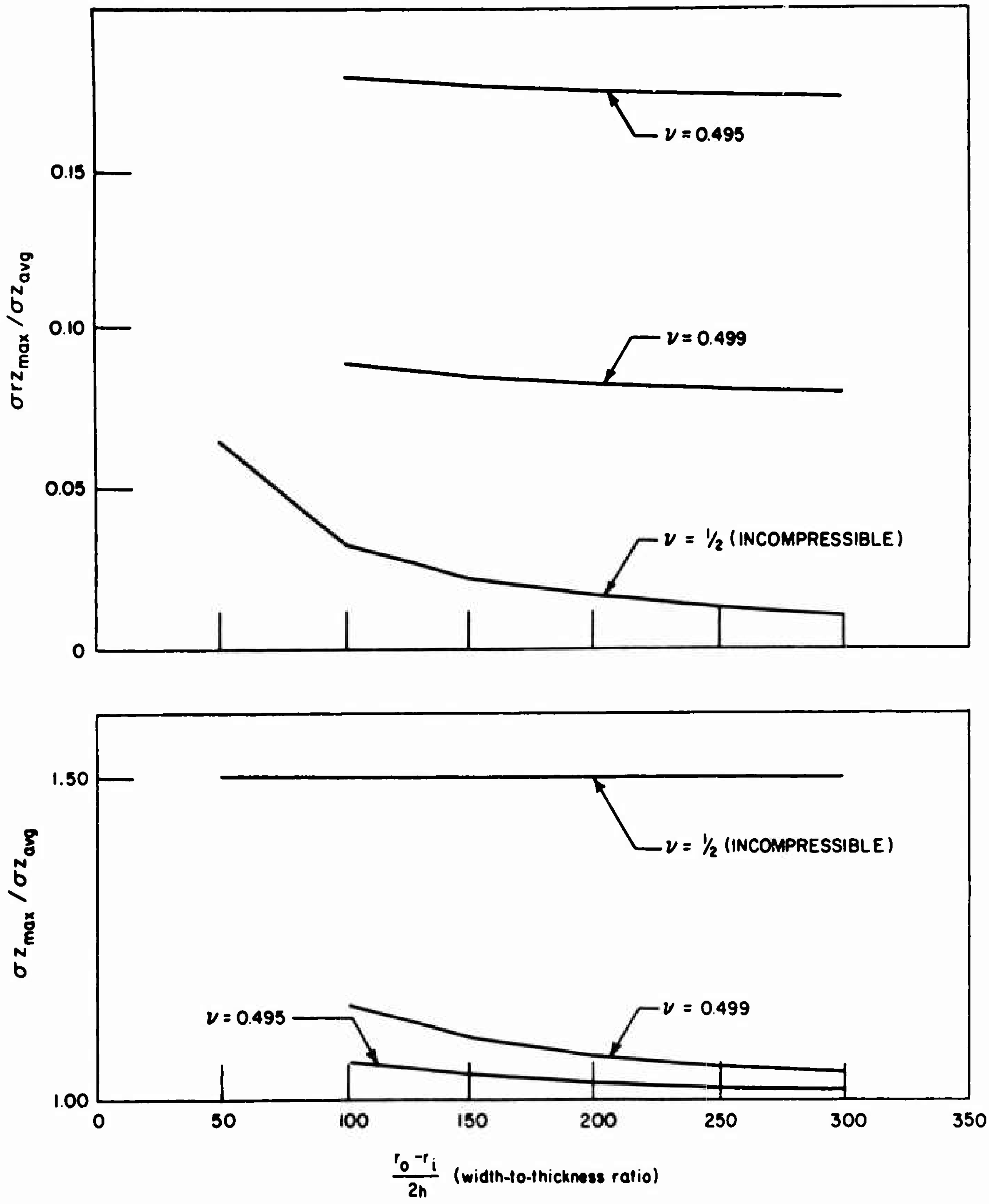


Figure 3. Average Pressure Versus Shear Strain (Incompressible Elasticity Theory).



NOTE: FOR $\nu = \frac{1}{2}$, THE CURVE REPRESENTS P_{max}/P_{avg} AT THE BOUNDARY OF ELASTOMER AND METAL LAMINA

Figure 4. Comparative Data for Compressible and Incompressible Theory.

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APPENDIX A

COMPUTER SOLUTION FOR A COMPRESSIBLE ELASTOMER

This is the formulation of the pure axial compression problem of one lamination, following linear (or classical) elasticity theory, that was prepared for computer solution using finite-difference techniques and iteration schemes. The geometry is that of the flat thrust bearing.

Consider a thin disk of elastomer compressed between two rigid plates and assume that the disk is composed of a compressible material with Poisson's ratio, ν . The shape and the dimension of the disk are shown in Figure 5.

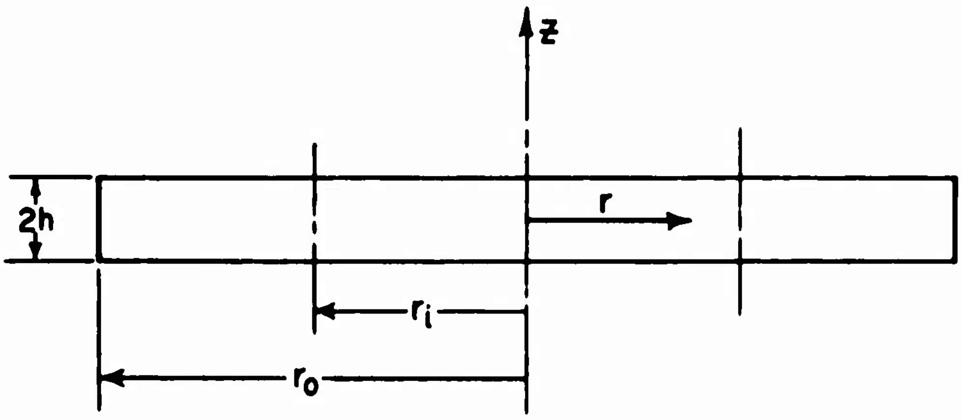


Figure 5. Sketch of Coordinates for One Lamination.

Since the disk is circular, it is convenient to use cylindrical coordinates (r, θ, z). The symmetry property of this problem indicates that stresses and displacements are functions of r, z only.

The equations of equilibrium are thus simplified to

$$\frac{\partial^2 w}{\partial r^2} + \frac{2(1-\nu)}{1-2\nu}\frac{\partial^2 w}{\partial z^2} + \frac{1}{1-2\nu}\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{1-2\nu}\frac{1}{r}\frac{\partial u}{\partial z} = 0 \qquad \text{(A-1a)}$$

and

$$\frac{1}{1-2\nu}\frac{\partial^2 w}{\partial r \partial z} + \frac{\partial^2 u}{\partial z^2} + \frac{2(1-\nu)}{1-2\nu}\frac{\partial^2 u}{\partial r^2} + \frac{2(1-\nu)}{1-2\nu}\frac{1}{r}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) = 0, \qquad \text{(A-1b)}$$

where u and w are the displacements in radial and axial directions, respectively.

The boundary conditions are

$$\begin{array}{lll} u = 0, & w = -\delta & \text{at } z = +h \\ u = 0, & w = \delta & \text{at } z = -h \\ \sigma_r = 0, & \sigma_{rz} = 0 & \text{at } r = r_o \text{ and } r = r_i . \end{array} \qquad \text{(A-2)}$$

Since the mid-plane (z = 0) is a plane of symmetry, the boundary conditions can be replaced on z = -h by the symmetry conditions at z = 0.

$$\text{At } z = o, \quad w = 0 \quad \text{and} \quad \frac{\partial u}{\partial z} = 0 ; \qquad \text{(A-3)}$$

thus, the problem can be solved only for the domain z = 0 to h and r = r_i to r_o.

Because of the complexity of this problem, a computer program is written to solve this set of simultaneous equations. The equations of equilibrium and the boundary conditions are written in finite-difference form as follows:

$$[1 - \frac{b}{2R(I)}] w_{i-1,j} + [1 + \frac{b}{2R(I)}] w_{i+1,j} + \frac{2(1-\nu)}{1-2\nu} w_{i,j-1}$$
$$+ \frac{2(1-\nu)}{1-2\nu} w_{i,j+1} + (\tfrac{1}{4})(\frac{1}{1-2\nu})(u_{i+1,j+1} - u_{i-1,j+1} + u_{i-1,j-1} - u_{i+1,j-1})$$
$$+ (\frac{b}{2R(I)})(\frac{1}{1-2\nu})(u_{i,j+1} - u_{i,j-1}) - 2(\frac{3-4\nu}{1-2\nu}) w_{i,j} = 0 \qquad \text{(A-4a)}$$

and

$$\tfrac{1}{4} (\frac{1}{1-2\nu})(w_{i+1,j+1} - w_{i-1,j+1} + w_{i-1,j-1} - w_{i-1,j+1}) + u_{i,j-1}$$
$$+ u_{i,j+1} + \frac{2(1-\nu)}{1-2\nu} (1 + \frac{b}{2R(I)}) u_{i+1,j} + \frac{2(1-\nu)}{1-2\nu} (1 - \frac{b}{2R(I)}) u_{i-1,j}$$
$$- 2 [\frac{3-4\nu}{1-2\nu} + \frac{b^2}{2R^2(I)} (\frac{2(1-\nu)}{1-2\nu})] u_{i,j} = 0. \qquad \text{(A-4b)}$$

For a general point (i, j) in the body, and boundary conditions

$$\left.\begin{aligned} u_{i,mmm} &= 0 \\ w_{i,mmm} &= -\delta \end{aligned}\right\} z = h$$

$$\left.\begin{aligned} w_{i,2} &= 0 \\ u_{i,3} &= u_{i,1} \end{aligned}\right\} z = 0, \tag{A-5}$$

$$\left.\begin{aligned} r = r_i \; &\begin{cases} \frac{2(1-\nu)}{1-2\nu}(u_{3,j} - u_{i,j}) + \frac{2\nu}{1-2\nu}\left(\frac{2b}{R_i}\right)u_{2,j} + \frac{2\nu}{1-2\nu}(w_{2,j+1} - w_{2,j-1}) = 0 \\ u_{2,j+1} - u_{2,j-1} = -(w_{3,j} - w_{1,j}) \end{cases} \\ r = r_o \; &\begin{cases} \frac{2(1-\nu)}{1-2\nu}(u_{nnn+1,j} - u_{nnn-1,j}) + \frac{2\nu}{1-2\nu}\left(\frac{2b}{R_o}\right)u_{nnn,j} \\ \qquad + \frac{2\nu}{1-2\nu}(w_{nnn,j+1} - w_{nnn,j-1}) = 0 \\ u_{nnn,j+1} - u_{nnn,j-1} = -(w_{nnn+1,j} - w_{nnn-1,j}) \end{cases} \end{aligned}\right\} \tag{A-6}$$

for points (i, j) on the boundary, where b is the grid size and

$$R(z) = r_i + i \times b.$$

It is noted that since the material is compressible, there is a region in the body where the material experiences only hydrostatic compression. This knowledge permits further reduction in the domain of the problem by combining the two end conditions into one. That is to say, the set of equations is solved from both ends ($r = r_i$ and $r = r_o$) simultaneously, and the condition of hydrostatic compression is used as a boundary condition.

Denote ends $r = r_i$ and $r = r_o$ by $k = 1$ and $k = 2$, respectively, and define coefficients as shown in expressions (A-7) through (A-9). Then simplify the set of equations to those that appear.

Coefficients for the computer program are

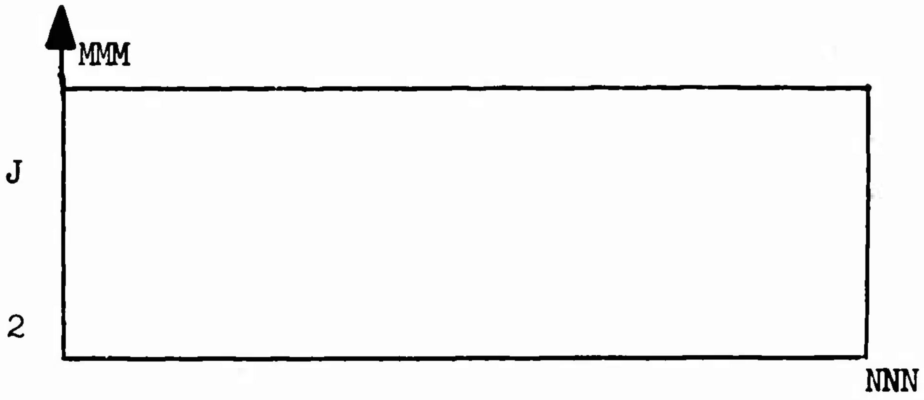
$$\left.\begin{array}{l} A(1) = 1-\nu \\ A(2) = \nu/(1-\nu) \\ A(3) = 3-4\nu \\ A(4) = 1-2\nu \\ A(5) = H/FM \\ A(6) = 1/(3-4\nu) \\ A(7) = A(4) \times A(6) \\ A(8) = A(1) \times A(6) \end{array}\right\} \qquad (A-7)$$

k=1 Rϕ(I,K) = RZERϕ + (FI-2.0) A(5)

$$\left.\begin{array}{l} C1(I,K) = A(5)/R\phi(I,K) \\ C2(I,K) = C1(I,K) ** 2 \\ C3(I,K) = 1.0 + 0.5 * C1(I,K) \\ C4(I,K) = 1.0 - 0.5 * C1(I,K) \\ C5(I,K) = A(3) + A(1) * C2(I,K) \\ C6(I,K) = 1.0/C5(I,K) \\ D1(K) \quad = + 1.0 \end{array}\right\} \qquad (A-8)$$

k=2

$$\left.\begin{array}{l} R\emptyset(I,K) = R\emptyset UT - (RI - 2.0) * A(5) \\ C1(I,K) = A(5)/R\emptyset(I,K) \\ C2(I,K) = C1(I,K) * C1(I,K) \\ C3(I,K) = 1.0 - 0.5 * C1(I,K) \\ C4(I,K) = 1.0 + 0.5 * C1(I,K) \\ C5(I,K) = A(3) + A(1) * C2(I,K) \\ C6(I,K) = 1.0/C5(I,K) \\ D1(K) \quad = - 1.0 \end{array}\right\}. \qquad (A-9)$$



Note that in the following expressions the double sign ± or ∓ appears. The upper sign refers to the case where k = 1, while the lower sign refers to that where k = 2.

Boundary conditions are

$$\left.\begin{aligned} w_{1,J} &= w_{3,J} \pm (u_{2,J+1} - u_{2,J-1}) \\ u_{1,J} &= u_{3,J} \pm \frac{\nu}{1-\nu} \left(\frac{2b}{R(I,K)} u_{2,J} + w_{2,J+1} - w_{2,J-1}\right. \end{aligned}\right\} . \tag{A-10}$$

Field equations are

$$\begin{aligned} w_{I,J} = \frac{1}{2(3+\frac{\lambda}{\mu})} \{ & [\mp \frac{b}{2*R\phi(I,K)}] w_{I-1,J} + [\pm \frac{b}{2*R\phi(I,K)}] w_{I+1,J} \\ & + (2 + \frac{\lambda}{\mu})(w_{I,J-1} + w_{I,J+1}) \pm \frac{1}{4} (1 + \frac{\lambda}{\mu})(u_{I+1,J+1} \\ & - u_{I-1,J+1} + u_{I-1,J-1} - u_{I+1,J-1}) + \frac{b}{2R\phi(I,K)} (1 + \frac{\lambda}{\mu}) \\ & (u_{I,J+1} - u_{I,J-1})\} \end{aligned} \tag{A-11a}$$

$$\begin{aligned} u_{I,J} = \frac{1}{2} [& \frac{1}{(3+\frac{\lambda}{\mu}) + (2+\frac{\lambda}{\mu})(\frac{b^2}{2R\phi^2(I,K)})}] \{\pm \frac{1}{4}(1+\frac{\lambda}{\mu})(w_{I+1,J+1} \\ & - w_{I-1,J+1} + w_{I-1,J-1} - w_{I+1,J-1}) + u_{I,J-1} + u_{I,J+1} \\ & + (2 + \frac{\lambda}{\mu}) [(1 \pm \frac{b}{2R\phi(I,K)}] u_{I+1,J} + (1 \mp \frac{b}{2R\phi(I,K)}) u_{I-1,J}\} . \end{aligned} \tag{A-11b}$$

Substituting $\lambda/\mu = 2\nu/1-2\nu$,

$$w_{I,J} = \frac{1-2\nu}{2(3-4\nu)} \{[1 \mp \frac{b}{2R\phi(I,K)}] w_{I-1,J} + [1 \pm \frac{b}{2R\phi(I,K)}] w_{I+1,J}\}$$

$$+ \frac{1-\nu}{3-4\nu} (w_{J,J-1} + w_{I,J+1}) \pm (\tfrac{1}{4}) \frac{1-\nu}{2(3-4\nu)} (u_{I+1,J+1} - u_{I-1,J+1}$$

$$+ u_{I-1,J-1} - u_{I+1,J-1}) + \frac{1-\nu}{2(3-4\nu)} (\frac{b}{2R\phi(I,K)}) (u_{I,J+1} - u_{I,J-1}) \qquad \text{(A-12a)}$$

$$u_{I,J} = \tfrac{1}{2} \{\frac{1}{[(3-4\nu) + (1-\nu) \frac{b^2}{R\phi^2(I,K)}}\} \{\pm\tfrac{1}{4} (w_{I+1,J+1} - w_{I-1,J+1}$$

$$+ w_{I-1,J-1} + w_{I+1,J-1}) + (1 - 2\nu)(u_{I,J-1} + u_{I,J+1})$$

$$+ 2(1-\nu) [(1 \pm \frac{b}{2R\phi(I,K)}) u_{I+1,J} + (1 \mp \frac{b}{2R\phi(I,K)}) u_{I-1,J}]\}. \qquad \text{(A-12b)}$$

From the definitions of the coefficients,

$$w_{I,J} = 0.5 * A(7) * \{C4(I,K) w_{I-1,J} + C3(I,K) w_{I+1,J}\}$$

$$+ A(8)* (w_{I,J-1} + w_{I,J+1}) + D1(K)* 0.125* A(8)* (u_{I+1,J+1}$$

$$- u_{I-1,J+1} + u_{I-1,J-1} - u_{I+1,J-1}) + 0.25* A(8)* C1(I,K)*$$

$$(u_{I,J+1} - u_{I,J-1}) \qquad \text{(A-13a)}$$

$$u_{I,J} = 0.5* C6(I,K)* D1(K)* 0.25* (w_{I+1,J+1} - w_{I-1,J+1} + w_{I-1,J-1}$$

$$- w_{I+1,J-1}) + A(4)* (u_{I,J-1} + u_{I,J+1}) + 2* A(1)$$

$$*(C3(I,K) u_{I+1,J} + C4(I,K) u_{I-1,J})). \qquad \text{(A-13b)}$$

Boundary conditions (end equation) are

$$w_{1,J} = w_{3,J} + D1(K)* (u_{2,J+1} - u_{2,J-1})$$

$$u_{1,J} = u_{3,J} + D1(K)* A(2)* (2* C1(2,K) u_{2,J} + w_{2,J+1} - w_{2,J-1}). \qquad \text{(A-14)}$$

Computation of stresses is as follows:

From the stress-strain relation

$$\sigma_{ij} = \lambda\Delta\delta_{ij} + 2G\epsilon_{ij}$$

and the strain-displacement relation, stresses can be calculated at any point in the body when the displacement field is known. Since the displacements are given by numerical values, finite-difference methods must be used to calculate strains.

At the top and bottom boundaries, the strains are

$$\epsilon_r = \frac{\partial u}{\partial r} = 0$$

$$\epsilon_\theta = \frac{u}{r} = 0$$

$$\epsilon_z = \frac{\partial w}{\partial z}$$

$$\epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)$$

and the stresses are

$$\sigma_z = (\lambda + 2G)\ \epsilon_z = \frac{2G(1 - \nu)}{1 - 2\nu}\ \frac{\partial w}{\partial z}$$

$$\sigma_{rz} = G\ \frac{\partial u}{\partial z}\ .$$

In order to evaluate the derivatives there, backward or forward differences must be used. At the top surface, backward difference is used.

$$\frac{\partial f}{\partial z} = \frac{1}{2\Delta z}\ (3f(I,J) - 4f(I,J-1) + f(I,J-2))$$

Since in the computer result u and w are nondimensionalized with respect to h, $\Delta z = h/m$.

For m = 4,

$$\frac{\partial w}{\partial z} = 2(3w\ (I,mmm) - 4w(I,m) + w(I,m-1))$$

$$\frac{\partial u}{\partial z} = 2(3u\ (I,mmm) - 4u(I,m) + u(I,m-1));$$

hence,

$$\sigma_z = \frac{4(1-\nu)}{1-2\nu}\ G(3w\ (I,mmm) - 4w(I,m) + w(I,m-1)$$

$$\sigma_{rz} = 2G(3u\ (I,mmm) - 4u(I,m) + u(I,m-1).$$

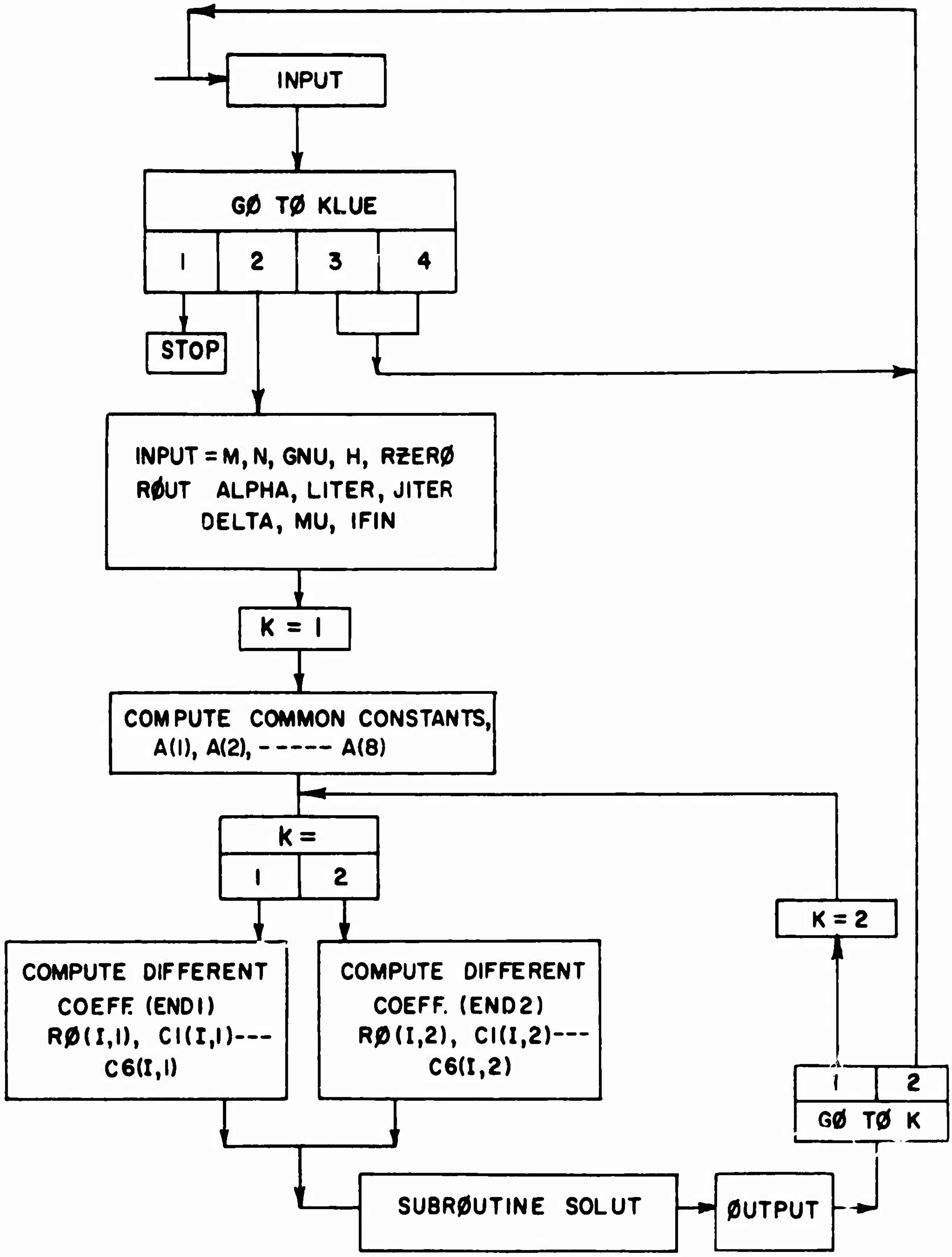


Figure 6. Main Flow Chart

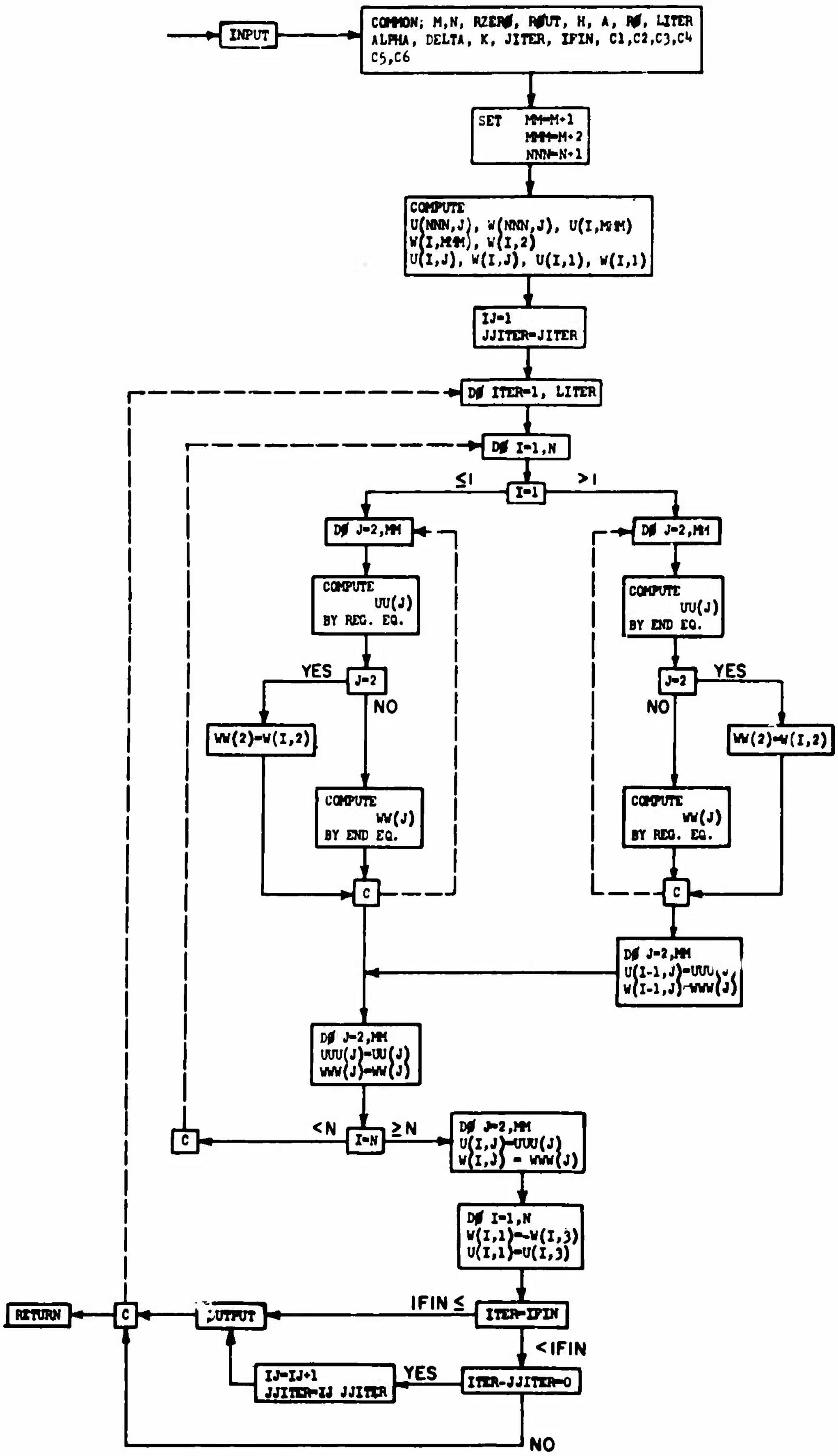


Figure 7. Subroutine Solut.

```
 LAMINATED BEARING

C
C COMPRESSIBLE MATERIAL
C
      DIMENSION  A(8), RO(401,2), C1(401,2), C2(401,2),  C3(401,2),
     1C4(401,2), C5(401,2), C6(401,2), D1(2)
      COMMON  M, RZERO, ROUT, H, A, RO, N, LITER, ALPHA, DELTA,
     1CONST, K, JITER, IFIN, C1, C2, C3, C4, C5, C6, D1
  110 READ INPUT TAPE 2,1, KLUE , ICLUE
    1 FORMAT (2I5)
      GO TO (500,100,400,400),KLUE
  100 READ INPUT TAPE 2, 2,    N, M, RZERO, ROUT, GNU, H   ,JITER,IFIN
    2 FORMAT (2I5,4E15.8/2I6)
      READ INPUT TAPE 2, 3, LITER, ALPHA, DELTA,         CONST, PMU
    3 FORMAT (I5, 4E15.8 )
      A(1) = 1.0 - GNU
      A(2) = GNU/A(1)
      A(3) = 3.0 - 4.0*GNU
      A(4) = 1.0 - 2.0*GNU
      FM = M
      A(5) = H/FM
      A(6) = 1.0/A(3)
      A(7) = A(4)*A(6)
      A(8) = A(1)*A(6)
      CALL PDUMP (A,A(8),1)
      K = 1
   20 CONTINUE
      GO TO (30,40),K
   30 NNN = N+2
      DO 32 I = 2,NNN
      FI = I
      RO(I,K) = RZERO + (FI - 2.0) * A(5)
      C1(I,K) = A(5)/RO(I,K)
      C2(I,K) = C1(I,K) * C1(I,K)
      C3(I,K) = 1.0 + (0.5 * C1(I,K))
      C4(I,K) = 1.0 - (0.5 * C1(I,K))
      C5(I,K) = A(3) + A(1)*C2(I,K)
      C6(I,K) = 1.0/C5(I,K)
   32 CONTINUE
      D1(K) = 1.0
      GO TO 50
   40 NNN = N+2
      DO 42 I=2,NNN
      FI = I
      RO(I,K) = ROUT - (FI-2.0)*A(5)
      C1(I,K) = A(5)/RO(I,K)
      C2(I,K) = C1(I,K) * C1(I,K)
      C3(I,K) = 1.0 - ( 0.5 * C1(I,K))
      C4(I,K) = 1.0 + ( 0.5 * C1(I,K))
      C5(I,K) = A(3) + A(1)*C2(I,K)
      C6(I,K) = 1.0/C5(I,K)
   42 CONTINUE
      D1(K) = -1.0
      GO TO 60
   50 WRITE OUTPUT TAPE 10,10, RZERO,ROUT
   10 FORMAT (1H1 3X 8HRZERO = F10.5, 3X 7HROUT = F10.5//)
```

```
      WRITE OUTPUT TAPE 10,11, GNU, M, H, N
   11 FORMAT (3X 6HGNU = F10.5, 3X 4HM = I5, 3X 4HH = F10.5, 3X 4HN =
     1I5//)
      WRITE OUTPUT TAPE 10,12, LITER,ALPHA,DELTA,       CONST,PMU
   12 FORMAT (1X 8HLITER = I5, 3X  8HALPHA = F10.5, 3X 8HDELTA = F10.5
     1, 3X 8HCONST = F10.5, 3X 6HPMU = F10.5////)
   60 CALL  SOLUT
      GO TO (89,108),K
   89 GO TO (108,90) , ICLUE
   90 K = 2
      GO TO 20
  108 CONTINUE
      GO TO 110
  400 CONTINUE
      GO TO 110
  500 CALL DUMP
      END(1,1,0,0,0,1,1,1,0,1,0,0,0,0,0)
```

04/20/64

```
      SUBROUTINE  SOLUT
      DIMENSION  A(8), RO(401,2), C1(401,2), C2(401,2),  C3(401,2),
     1C4(401,2), C5(401,2), C6(401,2), D1(2), U(401,10), W(401,10),
     1UUU(10), WWW(10),  UU(10), WW(10)
      COMMON  M, RZERO, ROUT, H, A, RO, N, LITER, ALPHA, DELTA,
     1CONST, K, JITER, IFIN, C1, C2, C3, C4, C5, C6, D1
      MM = M + 1
      MMM = M + 2
      NN = N+1
      NNN = N+2
      FM = M
      DO 10  J = 2,MMM
      U(NNN,J) = 0.0
      FJ = J
      W(NNN,J) = -(FJ-2.0)*DELTA/FM
   10 CONTINUE
      DO 15 I = 2,NN
      U(I,MMM) = 0.0
      W(I,MMM) = -DELTA
      W(I,2) = 0.0
   15 CONTINUE
      DO 20 I = 2,NN
      DO 20 J = 2,MM
      FJ = J
      U(I,J) = -D1(K)*CONST*DELTA*(1.0-((FJ-2.0)*A(5)/H)**2)
     1*EXPF(D1(K)*(RO(2,K)-RO(I,K)))
      W(I,J) = -(FJ-2.0)*DELTA/FM
   20 CONTINUE
      IJ = 1
      JJITER = JITER
      DO 110 ITER = 1,LITER
      DO 25 I = 2,NNN
      U(I,1) = U(I,3)
      W(I,1) = -W(I,3)
   25 CONTINUE
      DO 30 J = 2,MM
      U(1,J) = U(3,J)+D1(K)*A(2)*(2.0*C1(2,K)*U(2,J)+W(2,J+1)-W(2,J-1))
      W(1,J) = W(3,J)+D1(K)*(U(2,J+1)-U(2,J-1))
   30 CONTINUE
      U(1,1) = U(1,3)
      W(1,1) = -W(1,3)
      DO 80 I = 2,NN
      DO 50 J = 2,MM
      UEVAL = 0.5*C6(I,K)*(D1(K)*0.25*(W(I+1,J+1)-W(I-1,J+1)+W(I-1,J-1)
     1-W(I+1,J-1))+A(4)*(U(I,J-1)+U(I,J+1))+2.0*A(1)*(C3(I,K)*U(I+1,J)
     1+C4(I,K)*U(I-1,J)))
      UU(J) = ALPHA * UEVAL + (1.0-ALPHA) * U(I,J)
      IF (J-2) 45,40,45
   40 WW(2) = W(I,2)
      GO TO 50
   45 WEVAL = 0.5*A(7)*(C4(I,K)*W(I-1,J) + C3(I,K)*W(I+1,J))
     1+A(8)*(W(I,J-1)+W(I,J+1))+ D1(K)*0.125*A(8)*(U(I+1,J+1)
     1-U(I-1,J+1)+U(I-1,J-1)-U(I+1,J-1))+0.25*A(8)*C1(I,K)
     1*(U(I,J+1)-U(I,J-1))
      WW(J) = ALPHA * WEVAL + (1.0-ALPHA) * W(I,J)
   50 CONTINUE
```

```
      IF (I-2) 60,60,70
   60 DO 65 J = 2,MM
      UUU(J) = UU(J)
      WWW(J) = WW(J)
   65 CONTINUE
      GO TC 80
   70 DO 75 J=2,MM
      U(I-1,J) = UUU(J)
      W(I-1,J) = WWW(J)
   75 CONTINUE
      CO 79 J = 2,MM
      UUU(J) = UU(J)
      WWW(J) = WW(J)
   79 CONTINUE
   80 CONTINUE
      DO 85 J = 2,MM
      U(NN,J) = UUU(J)
      W(NN,J) = WWW(J)
   85 CONTINUE
      IF(ITER - IFIN) 101, 105, 105
  101 IF (ITER - JJITER) 110,104,104
  104 IJ = IJ+1
      JJITER = IJ*JITER
  105 INCR = 1
      JJ = 1
      KK = MMM - 6
  102 IF (KK)106,107,107
  106 JJJ= MMM
      GO TO 108
  107 JJJ = INCR * 6
  108 WRITE OUTPUT TAPE 10,1, ITER
    1 FORMAT (1H1 17HITERATION  NO. = I5///)
      WRITE OUTPUT TAPE 10,3
    3 FORMAT ( 7HU ARRAY//)
      WRITE OUTPUT TAPE 10,2,((U(I,J), J=JJ,JJJ),I=1,NNN)
      WRITE OUTPUT TAPE 10,4
    4 FORMAT (//7HW ARRAY//)
      WRITE OUTPUT TAPE 10,2,((W(I,J), J=JJ,JJJ),I=1,NNN)
    2 FORMAT (6(4X E15.8))
      IF (JJJ-MMM) 109,103,109
  109 KK = KK-6
      INCR = INCR + 1
      JJ = JJ+6
      GO TO 102
  103 CONTINUE
  110 CONTINUE
      RETURN
      END(1,1,0,0,0,1,1,1,0,1,0,0,0,0,0)
```

APPENDIX B

ANALYTICAL APPROXIMATION AND COMPUTER SOLUTION FOR AN INCOMPRESSIBLE ELASTOMER

Governing equations are again written in cylindrical coordinates, as shown in Appendix A. The equations of equilibrium remain the same, but the stress-strain relations become

$$\sigma_{ij} = P\,\delta_{ij} + 2G\epsilon_{ij} \tag{B-1}$$

where an additional unknown pressure, P, is introduced into the relation due to incompressibility. Furthermore, since there is no volume change,

$$\epsilon_{ii} = 0 \tag{B-2}$$

and strains are deviatoric. Taking into account the axial symmetry property, the nonvanishing strains are

$$\epsilon_r = \frac{\partial u}{\partial r}$$

$$\epsilon_\theta = \frac{u}{r}$$

$$\epsilon_z = \frac{\partial w}{\partial z}$$

$$\epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) . \tag{B-3}$$

Substituting the stress-strain relation and the strain-displacement relation into the equilibrium equation, we obtain the governing equation in terms of displacement.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{1}{G}\frac{\partial P}{\partial r} \quad \ldots . \tag{B-4a}$$

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} = -\frac{1}{G}\frac{\partial P}{\partial z} \quad \ldots . \tag{B-4b}$$

Since there are three unknowns, u, w, and P, three equations are needed; the third equation is furnished by equation (B-2),

$$\epsilon_{ii} = 0$$

or

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \; \ldots . \qquad \text{(B-4c)}$$

Equations (B-4a, b, and c) coupled with boundary conditions

$$\begin{array}{ll} u = 0 & z = \pm h \\ w = -w_o & z = + h \\ w = w_o & z = - h \\ \sigma_{rz} = 0 & r = r_i \text{ and } r_o \\ \sigma_{rr} = 0 & r = r_i \text{ and } r_o \end{array} \qquad \text{(B-5)}$$

completely define the problem. However, the deformations are extremely complex, owing to the imposed boundary conditions, and the mathematical analysis becomes intractable. Thus, the use of approximate methods and computer solutions becomes necessary.

APPROXIMATE SOLUTION

Because of the geometry of one lamination, $[h \ll (r_o - r_i)]$, the boundary conditions $\sigma_{rr} = 0$ and $\sigma_{rz} = 0$ may be replaced by an average condition.

Let

$$\left.\begin{array}{l} \sigma_{rr} = \dfrac{1}{2h} \displaystyle\int_{-h}^{h} \sigma_{rr} \, dz = 0 \\ \\ \sigma_{rz} = \dfrac{1}{2h} \displaystyle\int_{-h}^{h} \sigma_{rz} \, dz = 0 \end{array}\right\} ; \qquad \text{(B-6)}$$

that is to say, the system of forces acting on the boundary is replaced by an equivalent system. By Saint Venant's principle, this replacement will produce negligible influence at some distance away from the boundary.

Consider a solution of the form

$$u = (ar - br^{-1})\,(1 - z^2/h^2)\ \ldots \tag{B-7}$$

$$w = -2az\left(1 - \frac{z^2}{3h^2}\right)\ \ldots \tag{B-8}$$

where a and b are constants. Substituting equations (B-7) and (B-8) into the governing equations, it is found that equation (B-4c) is identically satisfied and that equations (B-4a) and (B-4b) yield

$$P/G = a\frac{r^2}{h^2} - \frac{2b}{h^2}\ln r + 2a\left(1 - \frac{z^2}{h^2}\right) + C_2\ \ldots \tag{B-9}$$

where C_2 is a constant.

This solution satisfies boundary conditions on $z = \pm h$, identically, provided

$$a = \frac{3}{4}\frac{w_o}{h}\,. \tag{B-10}$$

The average boundary condition on $r = r_i$ and r_o gives

$$\frac{2}{2h}\int_{-h}^{h}\frac{\partial u}{\partial r}\,dz + \frac{1}{2h}\int_{-h}^{h} P/G\,dz = 0\ \ldots \tag{B-11}$$

$$\frac{1}{2h}\int_{-h}^{h}\frac{\partial u}{\partial z}\,dz + \frac{1}{2h}\int_{-h}^{h}\frac{\partial w}{\partial r}\,dz = 0\ \ldots. \tag{B-12}$$

Substituting equations (B-7), (B-8), and (B-9) into (B-11) and (B-12), the last equation is satisfied identically and equation (B-11) yields

$$\frac{4}{3}(a + br^{-2}) + \frac{2}{h^2}\left(\frac{ar^2}{2} - b\ln r\right) + \frac{4a}{3} + C_2 = 0$$

or

$$\frac{ar^2}{h^2} - \frac{2b}{h^2}\ln r + \frac{4}{3}br^{-2} + \frac{8}{3}a + C_2 = 0\,. \tag{B-13}$$

Hence,

$$\left\{\begin{array}{l} a\dfrac{r_i^2}{h^2} - \dfrac{2b}{h^2}\ln r_i + \dfrac{4}{3}br_i^{-2} + \dfrac{8}{3}a + C_2 = 0 \\ \\ a\dfrac{r_o^2}{h^2} - \dfrac{2b}{h^2}\ln r_o + \dfrac{4}{3}br_o^{-2} + \dfrac{8}{3}a + C_2 = 0 \end{array}\right\} \tag{B-13a}$$

$$b = \frac{a\,h^2\left(\dfrac{r_o^2}{h^2} - \dfrac{r_i^2}{h^2}\right)}{2\ln r_o/r_i + \dfrac{4}{3}\left[\dfrac{h^2}{r_i^2} - \dfrac{h^2}{r_o^2}\right]} \tag{B-14}$$

$$C_2 = -\left[a\frac{r_i^2}{h^2} - \frac{2b}{h^2}\ln r_i + \frac{8}{3}a + \frac{4}{3}b\,r_i^{-2}\right]. \tag{B-15}$$

The pressures at the top and bottom boundaries (bonded surface) can be calculated from equation (B-9).

$$P/G\Big|_{z=\pm h} = a\frac{r^2}{h^2} - \frac{2b}{h^2}\ln r + C_2 \tag{B-16}$$

where a is substituted from (B-10) and b and C_2 from above.

The total axial force on the bearing is computed by integrating equation (B-16) over the surface of z = h (the bonded surface).

$$F/G = \int_{r_i}^{r_o} (P/G)\ r\ dr\ d\theta$$

$$= 2\pi \int_{r_i}^{r_o} \left[\frac{ar^3}{h^2} - \frac{2b}{h^2} r \ln r + C_2 r\right] dr$$

$$F/G = \pi \left[r^2 \left(\frac{ar^2}{2h^2} - \frac{2b}{h^2} \ln r + \frac{b}{h^2} + C_2\right) \right]_{r_i}^{r_o} . \qquad \text{(B-17)}$$

Average pressure on the bonded surface is the total force divided by the area, or

$$\frac{P_{ave}}{G} = \frac{1}{r_o^2 - r_i^2} \left[r^2 \left(\frac{ar^2}{2h^2} - \frac{2b}{h^2} \ln r + \frac{b}{h^2} + C_2\right) \right]_{r_i}^{r_o} . \qquad \text{(B-18)}$$

To find the maximum pressure on the bonded surface (z = h), radius is found at which pressure is maximum by differentiating equations (B-17) and then substituting that radius into equation (B-16).

$$\frac{1}{G}\frac{dP}{dr} = \frac{2ar}{h^2} - \frac{2b}{h^2}\frac{1}{r} = 0$$

and

$$r = \sqrt{\frac{b}{a}} \quad . \qquad \text{(B-19)}$$

Now

$$\left.\frac{P_{max}}{G}\right|_{z=h} = \frac{b}{h^2}\left[1 - 2 \ln\sqrt{b/a}\right] + C_2 . \qquad \text{(B-20)}$$

The stress distribution at the inner and outer radial boundaries is now approximated by

$$\frac{\sigma_r}{G} = 2\frac{\partial u}{\partial r} + \frac{P}{G}, \tag{B-21}$$

where from equation (B-7)

$$\frac{\partial u}{\partial r} = \left(a + \frac{b}{r^2}\right)\left(1 - \frac{z^2}{h^2}\right),$$

and by

$$\frac{\sigma_{rz}}{G} = \frac{\partial u}{\partial z}, \tag{B-22}$$

where from equation (B-7)

$$\frac{\partial u}{\partial z} = -\left(ar - \frac{b}{r}\right)\frac{2z}{h^2} .$$

Consider now a typical example:

r_i = 0.75 inch

r_o = 1.125 inches

h = 0.75 (10^{-3}) inch
(width-to-thickness ratio of 250).

If the ratio of axial compressive deformation to the elastomer thickness is chosen as

$$\frac{w_o}{h} = (10^{-3}),$$

then

$$a = 0.75\ (10^{-3})$$

$$b \approx 1.156\ (10^{-3})\ h$$

$$C_2 \approx -1.415\ (10^{-3}).$$

The pressure on the bonded surfaces can be computed from equation (B-9):

$$P/G\Big|_{z=h} = a\,\frac{r^2}{h^2} - \frac{2b}{h^2}\,\ln r + C_2$$

$$= 0.75(10^{-3})\,\frac{r^2}{h^2} - 2.312(10^{-3})\,\ln r - 1.415(10^{-3})$$

and

r	r/h	ln r	P
0.75"	10^3	-0.2877	0
0.825"	1.1 (10^3)	-0.1924	$0.0629(10^3)G$
0.90"	1.2 (10^3)	-0.1054	$0.0915(10^3)G$
0.975"	1.3 (10^3)	-0.0253	$0.0892(10^3)G$
1.05"	1.4 (10^3)	+0.0488	$0.058(10^3)G$
1.125"	1.5 (10^3)	+0.1178	0

Average pressure is computed from equation (B-18):

$$\frac{P_{ave}}{G}\Bigg|_{z=h} = \frac{1}{r_o^2 - r_i^2}\left[r^2\left(\frac{ar^2}{2h} - \frac{2b}{h^2}\,\ln r + \frac{b}{h^2} + C_2\right)\right]_{r_i}^{r_o}$$

$$= -\ 0.0629\ (10^{-3})$$

or, in magnitude,

$P_{ave} = 0.0629\ (10^{-3})G$ (psi when G is in units of psi).

The total axial force applied is the average pressure multiplied by the cross-sectional area of 2.2 inches^2.

$F = 0.1384\ (10^3)\ G$ (pounds when G is in units of psi)

Maximum pressure occurs from equation (B-19) at

$$r = \sqrt{b/a} = 0.931\ \text{inch}$$

and is of magnitude, from equation (B-20),

$$P_{max} = -0.0941\ (10^3)\ G\ (psi).$$

The ratio of $\frac{P_{max}}{P_{ave}}$ is 1.5.

Radial stress on the inner radial boundary is

$$\frac{\sigma_r}{G} = 2\frac{\partial u}{\partial r} + \frac{P}{G}$$

z/h	σ_r
0	$1.812\ (10^{-3})\ G$ (midway between bonded surface)
1/4	$1.427\ (10^{-3})\ G$
1/2	$0.485\ (10^{-3})\ G$
3/4	$-1.176\ (10^{-3})\ G$
1	$-3.5\ (10^{-3})\ G$.

Shear stress on the inner radial boundary is, from equation (B-22),

$$\frac{\sigma_{rz}}{G} = \frac{\partial u}{\partial z}$$

or

$$\sigma_{rz} = 0.812\ \frac{z}{h}\ G\ (psi).$$

This example is also a test of the approximation. From the above calculations for stress on the inner boundary, it can be seen that radial stress, σ_r, is of the order of (10^{-3} G); and with G = 100, for example, the radial stress is still of the order 0.1 psi. This is small as compared to the pressure developed within the bearing. Shear stress on the boundary varies from zero to 0.812 G. This is not small as compared to other shear stresses; but since a shear stress on this boundary does not influence greatly the pressure within the bearing, this is not a source of serious error. The approximation is considered to be very good for a wide, thin annulus.

COMPUTER SOLUTION

Due to symmetry with respect to z = 0, only half of the bearing need be considered; if z is positive, the symmetry condition then states

$$\left.\begin{aligned} & w(r,0) = 0 \\ & \left.\frac{\partial u}{\partial z}\right|_{z=0} = 0 \\ & \left.\frac{\partial P}{\partial z}\right|_{z=0} = 0 \\ & w(r,0^{+}) = -\,w(r,0^{-}) \end{aligned}\right\} . \qquad \text{(B-23)}$$

To solve this set of equations (B-4a, b, and c) by the finite-difference method, it is convenient to find an equation involving P explicitly. This can be achieved by differentiating between equations (B-4a) and (B-4b) rather than by using equation (B-4c). Thus,

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{\partial^2 P}{\partial z^2} = 0 \;\ldots. \qquad \text{(B-24)}$$

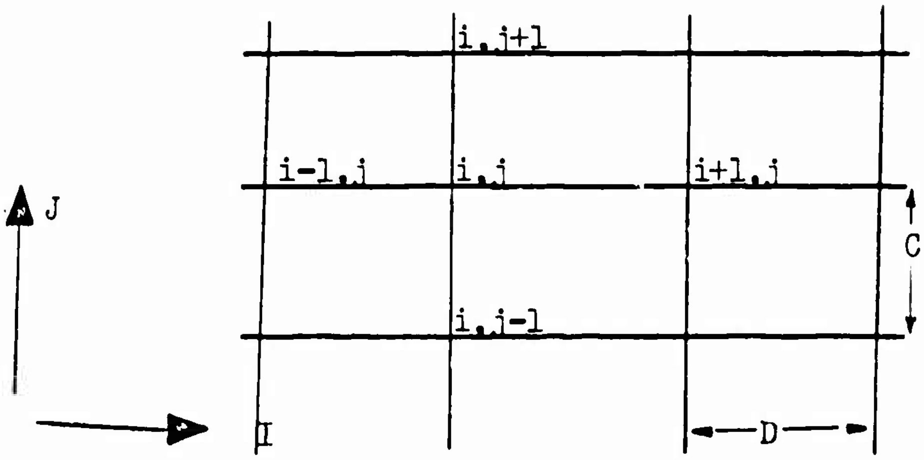
It is noted that equation (B-24) implies

$$\nabla^2 \left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}\right) = 0 \qquad \text{(B-25)}$$

rather than

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0. \qquad \text{(B-26)}$$

It is thus necessary to specify $\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$ on the boundary.



Written in finite-difference form, equations (B-4a), (B-4b), and (B-24) become

$$\left[2\left(1+\frac{D^2}{C^2}\right)+\frac{D^2}{R(I)}\right]u_{i,j} = (u_{i+1,j}+u_{i-1,j}) + \frac{D}{2R(I)}(_{i+1,j} - U_{i-1,j})$$

$$\frac{D^2}{C^2}(u_{i,j+1}+u_{i,j-1}) + \frac{D}{2H}(P_{i+1,j} - P_{i-1,j}) \qquad \text{(B-27a)}$$

$$2\left(1+\frac{D^2}{C^2}\right)w_{i,j} = (w_{i+1,j}+w_{i-1,j}) + \frac{D}{2R(I)}(w_{i+1,j} - w_{i-1,j})$$

$$+\frac{D^2}{C^2}(w_{i,j+1}+w_{i,j-1}) + \frac{D^2}{2C,H}(P_{i,j+1} - P_{i,j-1}) \qquad \text{(B-27b)}$$

$$2\left(1+\frac{D^2}{C^2}\right)P_{i,j} = (P_{i+1,j}+P_{i-1,j}) + \frac{D}{2R(I)}(P_{i+1,j} - P_{i-1,j})$$

$$+\frac{D^2}{C^2}(P_{i,j+1}+P_{i,j-1}) \qquad \text{(B-27c)}$$

where

$P = P/G$; $U_{i,j} = \frac{u}{H}$; $w_{i,j} = \frac{W}{H}$; and

$H = h$.

Treatment of equations at boundary:

Since the boundary conditions involve derivatives, fictitious points are introduced outside. The values of those points are found from boundary conditions.

At $r = r_i$

$$\left.\begin{aligned} w_{1,j} &= w_{3,j} - u_{2,j+1} + u_{2,j-1} \\ P_{2,j} &= u_{1,j} - u_{3,j} \\ u_{1,j} &= u_{3,j} + \frac{2}{R(Z)} u_{2,j} + \frac{D}{C} (w_{2,j+1} - w_{2,j-1}) \end{aligned}\right\}. \quad \text{(B-28)}$$

At $r = r_o$

$$\left.\begin{aligned} w_{NNN,j} &= w_{NE,j} + u_{NN,j+1} - u_{NN,j-1} \\ P_{NN,j} &= u_{NE,j} - u_{NNN,j} \\ u_{NNN,j} &= u_{NE,j} - \frac{R}{R(NN)} u_{NN,j} - \frac{D}{C} (w_{NN,j+1} - w_{NN,j-1}) \end{aligned}\right\}. \quad \text{(B-29)}$$

These equations, coupled with field equations (B-4a) and (B-4b), are enough to solve for the values at boundaries r_i, r_o and the fictitious points. However, it is not desirable to indroduce any fictitious points with P as unknown. Hence, $\frac{\partial P}{\partial r}$ is written in terms of forward or backward differences, and equation (B-4a) is modified accordingly.

$$\frac{\partial P}{\partial r} = \frac{1}{2D} (r\, P_{i+1,j} - 3\, P_{i,j} - P_{i+2,j}) \quad \text{(forward difference).} \quad \text{(B-30)}$$

$$\frac{\partial P}{\partial r} = \frac{1}{2D} (3\, P_{i,j} - 4\, P_{i-1,j} + P_{i-2,j}) \quad \text{(backward difference).} \quad \text{(B-31)}$$

At $z = h$, u and w are given but P is found through equation (B-24). In order to achieve this, it is necessary to introduce a fictitious point for P. The value of P at this point is found by using equations (B-4c) and (B-4b).

This set of finite-difference equations is solved by using the successive relaxation method. Values of u, w, and P at one point are calculated from values at neighboring points; they are u^{eva}, w^{eva}, and P^{eva}. New values of this point are obtained from u^{eva}, etc., through

$$u^{new} = \alpha\, u^{eva} + (1 - \alpha)\, u^{old}.$$

These new values can then be substituted for the old values immediately; then one can proceed to the next point.

Note on computer data:

Owing to the limitation of computer memory, the dimension of the bearing has been changed to

$$\begin{aligned} r_i &= 0.75 \text{ inch} \\ (r_o - r_i)/2h &= 50 \\ h &= 1 \times 10^{-3} \text{ inches}. \end{aligned}$$

The approximate solution is used as the first guess.

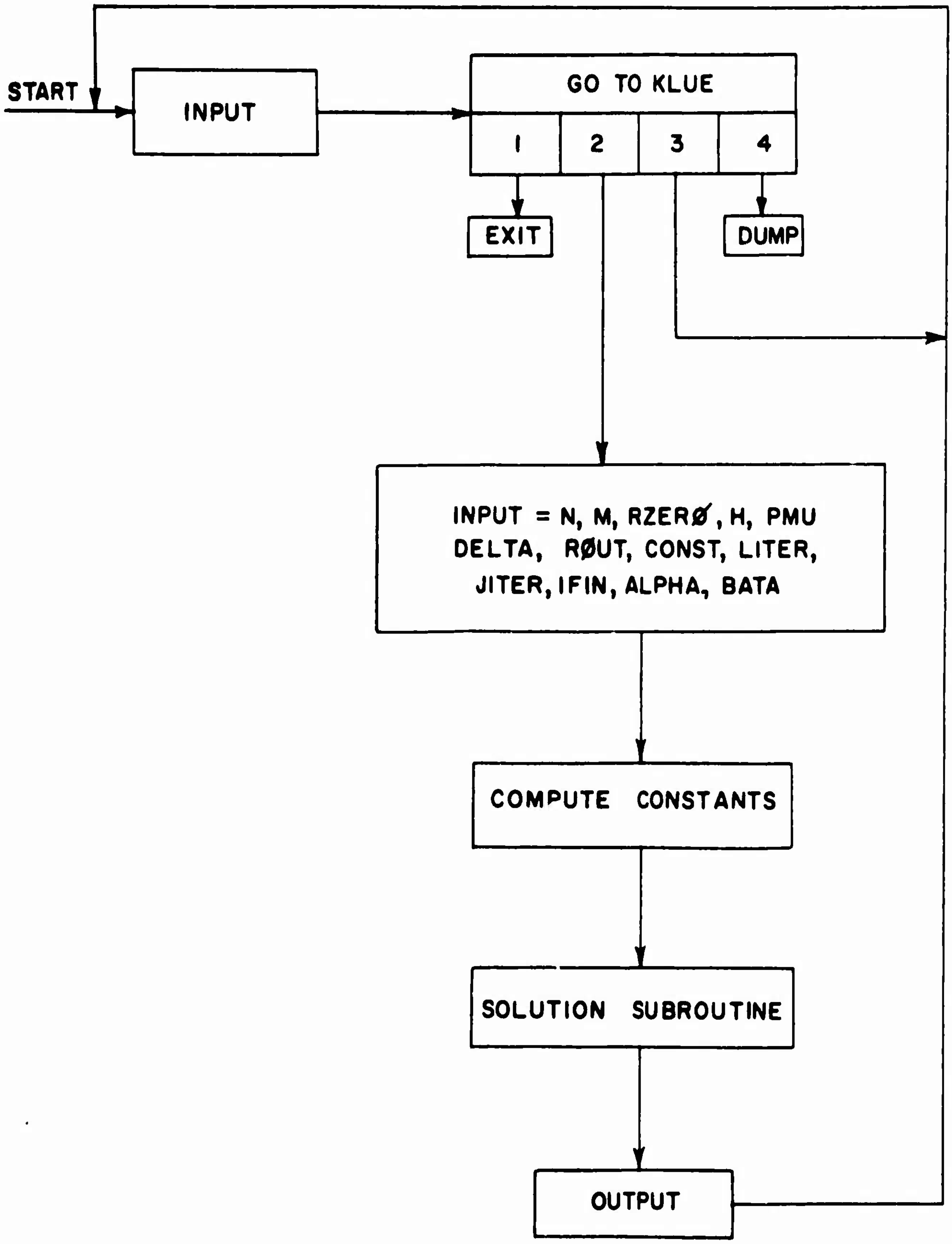


Figure 8. Main Flow Chart.

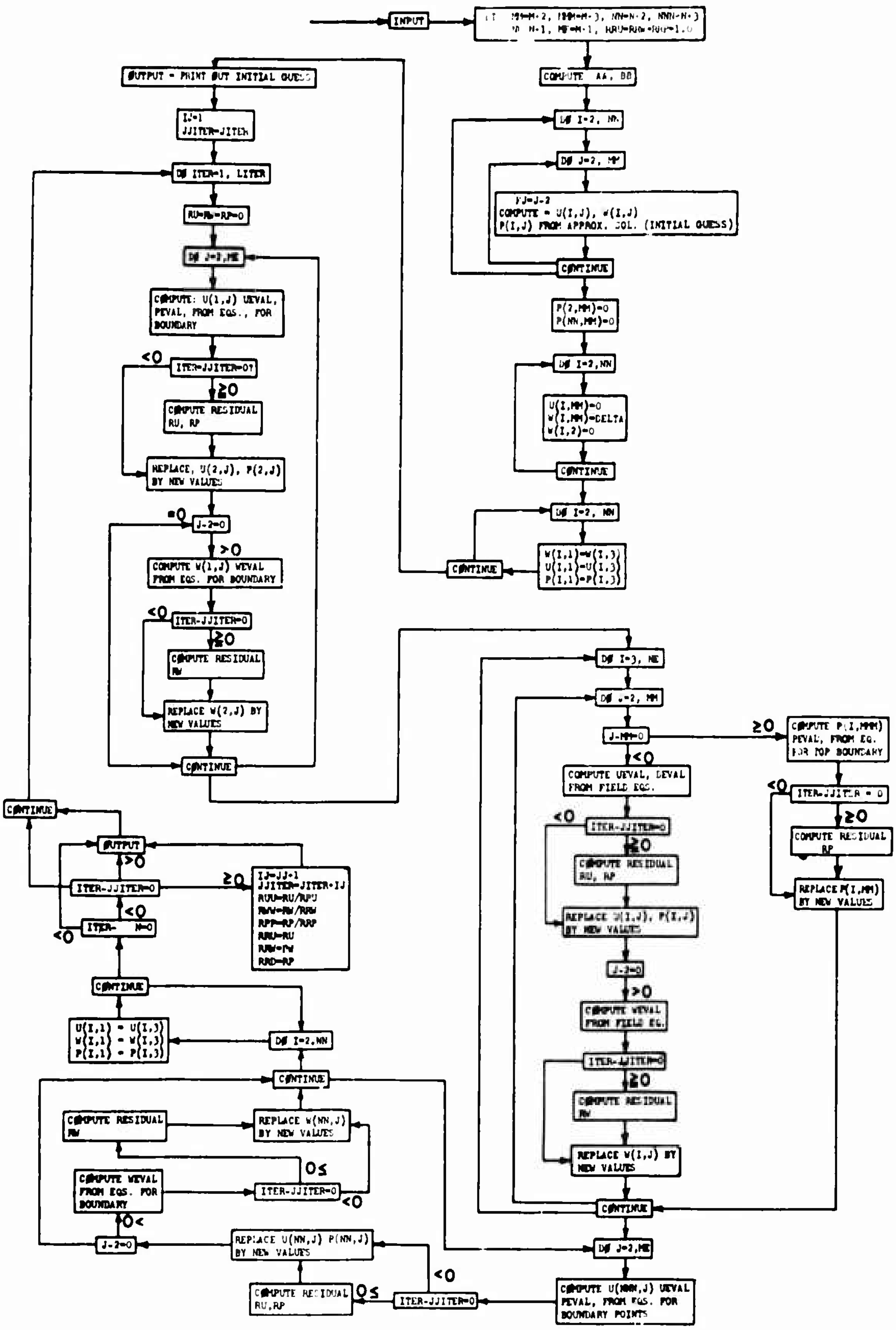


Figure 9. Solution Subroutine Flow Chart.

LAMINATED BEARING

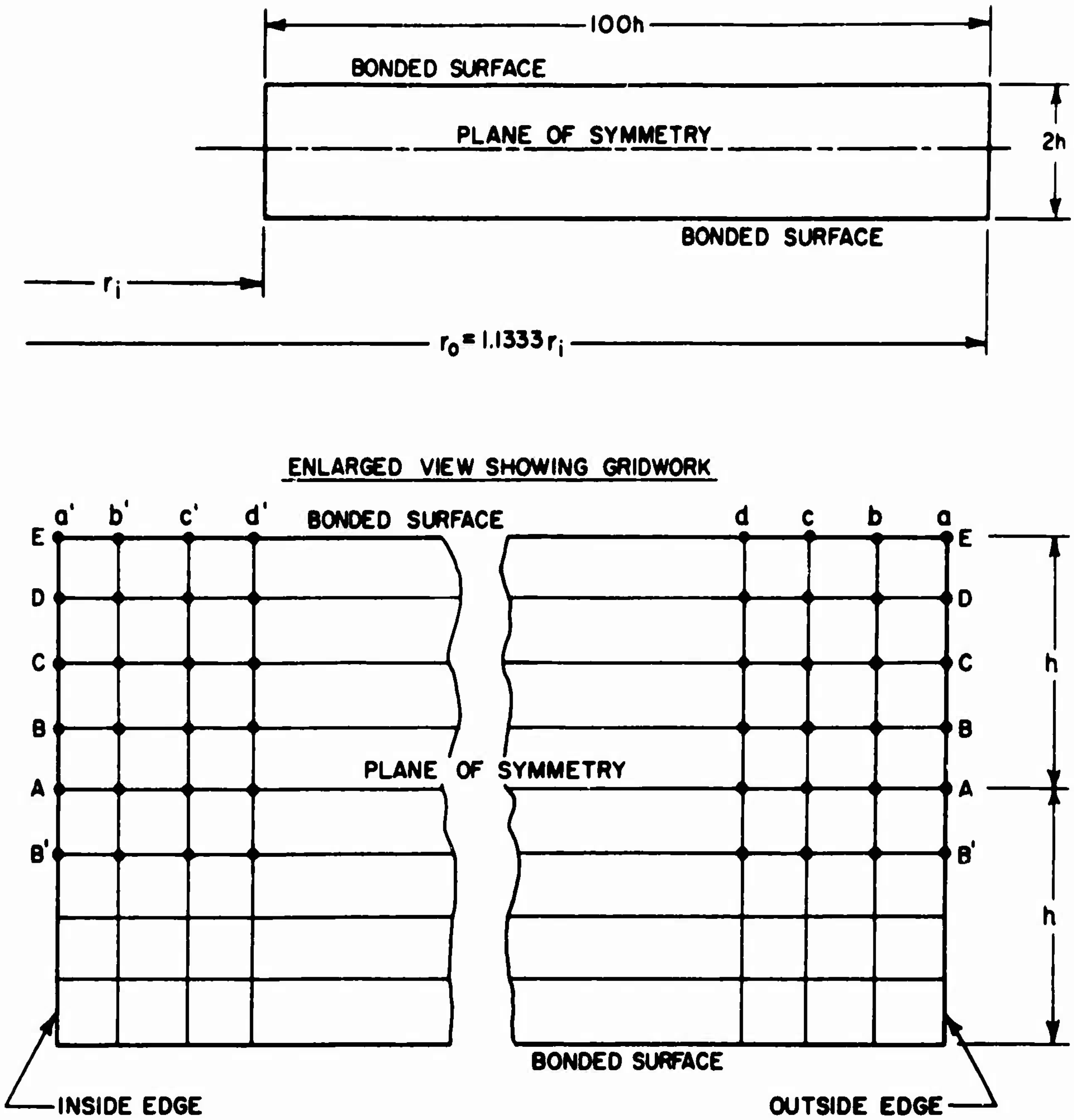
```
C
C  INCOMPRESSIBLE MATERIAL
C
      DIMENSION RO(500), B(8), AR1(500), AR2(500), AR3(500), AR4(500)
      COMMON  M, RZERO, H, RO, N, LITER, ALPHA, DELTA, BATA,ROUT,  B,C,D
     1,IFIN,JITER,AR1,AR2,AR3,AR4 ,CONST
  110 READ INPUT TAPE 2,1, KLUE
    1 FORMAT (I5)
      GO TO (500,100,300,400),KLUE
  100 READ INPUT TAPE 2,2,    N, M, RZERO, H, PMU, DELTA,ROUT,CONST
    2 FORMAT (2I5,4E15.8/2E15.8)
      READ INPUT TAPE 2,3,    LITER, JITER, IFIN, ALPHA, BATA
    3 FORMAT (3I5,2E15.8)
      WRITE OUTPUT TAPE 10,10, RZERO, H, PMU, DELTA,ROUT,CONST
   10 FORMAT (1H1 3X 8HRZERO = F10.5, 3X 4HH = F10.5,3X 6HPMU = F10.5,
     13X 8HDELTA = F10.5/3X 7HROUT = F10.2,3X 8HCONST = F10.5//)
      WRITE OUTPUT TAPE 10,11, N, M, LITER, JITER, IFIN, BATA, ALPHA
   11 FORMAT (1X 4HN = I5,3X 4HM = I5, 3X 8HLITER = I5, 3X 8HJITER = I5/
     13X 7HIFIN =  I5, 3X 7HBATA = F10.5, 3X 8HALPHA = F10.5////)
      FM = M
      NN = N+2
      C = H/FM
      D = C*BATA
      B(1) = D/C
      B(2) = B(1)**2
      B(3) = 1.0 + B(2)
      B(4) = 1.0/C
      B(5) = 1.0/D
      B(6) = C*B(1)
      B(7) = B(4)*B(1)
      B(8) = 1.0/B(3)
      DO 150 I = 2,NN
      FI = I - 2
      RO(I) = RZERO + FI*D
      AR1(I) = 1.0/[illegible]I)
      AR2(I) = D*A[illegible](I)
      AR3(I) = 2.0*B(3) + AR2(I)**2
      AR4(I) = 1.0/AR3(I)
  150 CONTINUE
      CALL PDUMP (RO,RO(10) ,1,B,B(8),1,AR1,AR1(10),1,AR2,AR2(10),
     11,AR3,AR3(10),1,AR4,AR4(10),1)
      CALL SOLUT
      GO TO 110
  500 CALL EXIT
  300 CONTINUE
  400 CALL DUMP
      END(1,1,0,0,0,1,1,1,0,1,0,0,0,0,0)
```

```
      SUBROUTINE  SOLUT
      DIMENSION  RO(500), U(500,13),W(500,13), P(500,13)
     1,B(8),AR1(500),AR2(500),AR3(500),AR4(500)
      COMMON  M, RZERO, H, RO, N, LITER, ALPHA, DELTA, BATA,ROUT,  B,C,D
     1,IFIN,JITER,AR1,AR2,AR3,AR4 ,CONST
      MM = M+2
      NN = N+2
      MMM = M+3
      NNN = N+3
      NE = N+1
      ME = M+1
      RRU = 1.0
      RRW = 1.0
      RRP = 1.0
      AA = 0.75*DELTA
      BB= AA*(ROUT**2 - RZERO**2) / (2.0*LOGF(ROUT/RZERO) -(4.0/3.0)*
     1((1.0 /ROUT)**2 - (1.0 /RZERO)**2))
      DO 10 I = 2,NN
      DO 10 J=2,MM
      FJ = J - 2
      U(I,J) = (AA * RO(I) - BB * AR1(I))* (1.0 -(FJ*C)**2)
      W(I,J) = - 2.0 * AA * (FJ*C) * (1.0 - (FJ*C)**2/3.0)
      P(I,J) = AA* (RO(I)**2 - RZERO**2)+2.0*BB* LOGF(RZERO*AR1(I))
     1+2.0 * AA* (1.0 - (FJ*C)**2) - 8.0*AA/3.0-4.0*BB/(3.0*RZERO**2)
   10 CONTINUE
      P(2,MM) = 0.0
      P(NN,MM) = 0.0
      DO 20 I = 2,NN
      U(I,MM) = 0.0
      W(I,MM) = -DELTA
      W(I,2)  = 0.0
   20 CONTINUE
      DO 30 I = 2,NN
      W(I,1) = -W(I,3)
      U(I,1) = U(I,3)
      P(I,1) = P(I,3)
   30 CONTINUE
      INCR = 1
      JJ=1
      KK=MM-6
 1128 IF (KK) 1130,1131,1131
 1130 JJJ=MM
      GO TO 1151
 1131 JJJ=INCR * 6
 1151 WRITE OUTPUT TAPE 10,3
      WRITE OUTPUT TAPE 10,2,((U(I,J), J=JJ,JJJ),I=1,NNN)
      WRITE OUTPUT TAPE 10,4
      WRITE OUTPUT TAPE 10,2,((W(I,J), J=JJ,JJJ),I=1,NNN)
      WRITE OUTPUT TAPE 10,5
      WRITE OUTPUT TAPE 10,2,((P(I,J),J=JJ,JJJ),I=1,NNN)
      IF (JJJ-MM) 1155,1160,1155
 1155 KK=KK-6
      INCR = INCR + 1
      JJ = JJ+6
      GO TO 1128
 1160 CONTINUE
```

```
      IJ = 1
      JJITER = JITER
      DO 200 ITER = 1,LITER
      RU = 0.0
      RW = 0.0
      RP = 0.0
      DO 35 J = 2,ME
      U(1,J) = U(3,J) + 2.0*AR2(2)*U(2,J) + B(1)*(W(2,J+1)-W(2,J-1))
      UEVAL  = AR4(2)*((U(3,J) + U(1,J)) + 0.5*AR2(2)*(U(3,J)-U(1,J))
     1+ B(2)*(U(2,J+1) + U(2,J-1)) + 0.5*D*(4.0*P(3,J) - 3.0*P(2,J)
     1-P(4,J)))
      PEVAL  = B(5)*(U(1,J) -U(3,J))
      IF ( ITER - JJITER) 34,29,29
   29 UU = ABSF(UEVAL - U(2,J))
      PP = ABSF(PEVAL - P(2,J))
      IF (RU - UU) 31,32,32
   31 RU = UU
   32 IF (RP - PP) 33,34,34
   33 RP = PP
   34 U(2,J)= ALPHA * UEVAL       + (1.0 - ALPHA) * U(2,J)
      P(2,J)= ALPHA * PEVAL       + (1.0 - ALPHA) * P(2,J)
      IF(J-2) 45,40,45
   40 GO TO 35
   45 W(1,J) =W(3,J)+B(1)*(U(2,J+1)-U(2,J-1))
      WEVAL = 0.5* B(8)           *((W(3,J)+W(1,J))+0.5*AR2(2)
     1*(W(3,J)-W(1,J))+B(2)      *(W(2,J+1)+W(2,J-1))+0.5*B(6)
     1*(P(2,J+1)-P(2,J-1)))
      IF ( ITER - JJITER) 38,36,36
   36 WW = ABSF(WEVAL - W(2,J))
      IF (RW - WW) 37,38,38
   37 RW = WW
   38 W(2,J)= ALPHA * WEVAL       + (1.0 - ALPHA) * W(2,J)
   35 CONTINUE
      DO 100 I = 3,NE
      DO 80  J = 2,MM
      IF(J - MM) 60,65,65
   60 UEVAL = AR4(I) * ((U(I+1,J)+U(I-1,J))+0.5*AR2(I)*(U(I+1,J)
     1-U(I-1,J))+B(2)*(U(I,J+1)+U(I,J-1))+0.5*D*(P(I+1,J)-P(I-1,J)))
      PEVAL = 0.5*B(8)*((P(I+1,J)+P(I-1,J))+0.5*AR2(I)*(P(I+1,J)
     1-P(I-1,J))+B(2)*(P(I,J+1)+P(I,J-1)))
      IF ( ITER - JJITER) 64,59,59
   59 UU = ABSF(UEVAL - U(I,J))
      PP = ABSF(PEVAL - P(I,J))
      IF (RU - UU) 61,62,62
   61 RU = UU
   62 IF (RP - PP) 63,64,64
   63 RP = PP
   64 U(I,J)= ALPHA * UEVAL       + (1.0 - ALPHA) * U(I,J)
      P(I,J)= ALPHA * PEVAL       + (1.0 - ALPHA) * P(I,J)
      IF (J-2) 75,70,75
   70 GO TO 80
   75 WEVAL = 0.5*B(8)*((W(I+1,J)+W(I-1,J))+0.5*AR2(I)*(W(I+1,J)
     1-W(I-1,J))+B(2)*(W(I,J+1)+W(I,J-1))+0.5*B(6)*(P(I,J+1)-P(I,J-1)))
      IF ( ITER - JJITER) 77,74,74
   74 WW = ABSF(WEVAL - W(I,J))
      IF (RW - WW) 76,77,77
```

```
 76 RW = WW
 77 W(I,J)= ALPHA * WEVAL      + (1.0 - ALPHA) * W(I,J)
    GO TO 80
 65 P(I,MMM) =P(I,ME)+4.0*B(4)*(W(I,MM)-W(I,ME))
    PEVAL = 0.5*B(8)*((P(I+1,MM)+P(I-1,MM))+0.5*AR2(I)*(P(I+1,MM)-
   1P(I-1,MM))+ B(2)*(P(I,MMM)+P(I,ME)))
    IF ( ITER - JJITER) 79,81,81
 81 PP = ABSF(PEVAL - P(I,J))
    IF (RP - PP) 78,79,79
 78 RP = PP
 79 P(I,MM) = ALPHA * PEVAL + (1.0 - ALPHA) * P(I,MM)
 80 CONTINUE
100 CONTINUE
    DO 95 J=2,ME
    U(NNN,J) = U(NE,J) - 2.0*AR2(NN)*U(NN,J) - B(1)*(W(NN,J+1)
   1-W(NN,J-1))
    UEVAL   = AR4(NN)*((U(NNN,J) + U(NE,J)) + 0.5*AR2(NN)*(U(NNN,J)
   1-U(NE,J)) + B(2)*(U(NN,J+1) + U(NN,J-1)) + 0.5*D*( 3.0*P(NN,J) -
   14.0*P(NE,J) + P(N,J)))
    PEVAL = B(5)*(U(NE,J) - U(NNN,J))
    IF ( ITER - JJITER) 85,86,86
 86 UU = ABSF(UEVAL-U(NN,J))
    IF (RU - UU) 82,83,83
 82 RU = UU
 83 PP = ABSF(PEVAL-P(NN,J))
    IF (RP - PP) 84,85,85
 84 RP = PP
 85 U(NN,J) = ALPHA * UEVAL + (1.0 - ALPHA) * U(NN,J)
    P(NN,J) = ALPHA * PEVAL + (1.0 - ALPHA) * P(NN,J)
    IF (J-2) 88,87,88
 87 GO TO 95
 88 W(NNN,J) =W(NE,J)-B(1)*(U(NN,J+1)-U(NN,J-1))
    WEVAL = 0.5*B(8)*((W(NNN,J)+W(NE,J))+0.5*AR2(NN)
   1*(W(NNN,J)-W(NE,J))+B(2)     *(W(NN,J+1)+W(NN,J-1))+0.5*B(6)
   1*(P(NN,J+1)-P(NN,J-1)))
    IF ( ITER - JJITER) 93,89,89
 89 WW = ABSF(WEVAL-W(NN,J))
    IF (RW - WW) 91,93,93
 91 RW = WW
 93 W(NN,J) = ALPHA * WEVAL + (1.0 - ALPHA) * W(NN,J)
 95 CONTINUE
    DO 110 I = 2,NN
    U(I,1) = U(I,3)
    W(I,1) = -W(I,3)
    P(I,1) = P(I,3)
110 CONTINUE
    IF (ITER-IFIN) 120,125,125
120 IF (ITER-JJITER) 200,135,135
135 IJ=IJ+1
    JJITER = IJ*JITER
    RUU = RU/RRU
    RWW = RW/RRW
    RPP = RP/RRP
    RRU = RU
    RRW = RW
    RRP = RP
```

```
125 INCR = 1
    JJ = 1
    KK = MM  - 6
128 IF (KK) 130,131,131
130 JJJ= MM
    GO TO 150
131 JJJ = INCR * 6
150 WRITE OUTPUT TAPE 10,1, ITER
  1 FORMAT (1H1 17HITERATION  NO. = I5///)
    WRITE OUTPUT TAPE 10,3
  3 FORMAT ( 7HU ARRAY//)
    WRITE OUTPUT TAPE 10,2,((U(I,J), J=JJ,JJJ),I=1,NNN)
    WRITE OUTPUT TAPE 10,4
  4 FORMAT (//7HW ARRAY//)
    WRITE OUTPUT TAPE 10,2,((W(I,J), J=JJ,JJJ),I=1,NNN)
    WRITE OUTPUT TAPE 10,5
  5 FORMAT (//7HP ARRAY//)
    WRITE OUTPUT TAPE 10,2,((P(I,J),J=JJ,JJJ),I=1,NNN)
  2 FORMAT (6(4X E15.8))
    WRITE OUTPUT TAPE 10,6, RU, RUU, RW, RWW, RP, RPP
  6 FORMAT (//2X 5HRU = E12.6, 2X 6HRUU = E12.6, 2X 5HRW = E12.6,
   12X 6HRWW = E12.6, 2X 5HRP = E12.6, 2X 6HRPP = E12.6)
    IF (JJJ - MM) 155,160,155
155 KK = KK-6
    INCR = INCR + 1
    JJ = JJ+6
    GO TO 128
160 CONTINUE
200 CONTINUE
    RETURN
    END(1,1,0,0,0,1,1,1,0,1,0,0,0,0,0)
```



Notes

1. Upper case letters on grid, Figure 8 refer to columns on computer output.
2. Lower case letters refer to row from either end as noted.
3. U array denotes radial displacement.
   example: 0.73755702 E - 01 means μ= 0.073755h
4. W array denotes axial displacement.
   example: 0.43044931 E - 02 means w = 0.004304h
5. P array denotes pressure.
   example: 0.13768537 E - 00 means P = 0.13768G

Figure 10. Geometry and Key to Computer Output.

ITERATION NO. = 600

U ARRAY

	Column B'	Column A	Column B	Column C	Column D	Column E
Inside Edge	0.	-0.41229375E-01	-0.43602026E-01	-0.40751156E-01	-0.64828350E-01	0.
Row a'	-0.71237648E-01	-0.76150560E-01	-0.71237648E-01	-0.56640506E-01	-0.32501135E-01	0.
Row b'	-0.71714811E-01	-0.76726006E-01	-0.71714811E-01	-0.56879040E-01	-0.32507309E-01	0.
Row c'	-0.69078283E-01	-0.73725202E-01	-0.69078263E-01	-0.55175163E-01	-0.32107724E-01	0.
Row d'	-0.67316884E-01	-0.71809946E-01	-0.67316884E-01	-0.53843167E-01	-0.31400554E-01	0.
	-0.65790301E-01	-0.70177127E-01	-0.65790301E-01	-0.52630898E-01	-0.30700341E-01	0.
	-0.64309352E-01	-0.68596926E-01	-0.64309352E-01	-0.51447152E-01	-0.30010595E-01	0.
	-0.62835054E-01	-0.67024239E-01	-0.62835054E-01	-0.50267086E-01	-0.29322827E-01	0.
	-0.61359814E-01	-0.65450599E-01	-0.61359814E-01	-0.49087747E-01	-0.28634455E-01	0.
	-0.59882776E-01	-0.63875055E-01	-0.59882776E-01	-0.47906150E-01	-0.27945212E-01	0.
	-0.58404551E-01	-0.62298249E-01	-0.58404551E-01	-0.46723595E-01	-0.27255405E-01	0.
	-0.56926022E-01	-0.60721120E-01	-0.56926022E-01	-0.45540744E-01	-0.26565450E-01	0.
	-0.55448083E-01	-0.59144630E-01	-0.55448083E-01	-0.44358461E-01	-0.25875766E-01	0.
	-0.53971490E-01	-0.57569576E-01	-0.53971490E-01	-0.43177202E-01	-0.25186707E-01	0.
	-C.52496862E-01	-0.55996624E-01	-0.52496862E-01	-0.41997511E-01	-0.24498561E-01	0.
	-0.51024680E-01	-0.54426206E-01	-0.51024680E-01	-C.40819773E-01	-0.23811550E-01	0.
	-0.49555320E-01	-0.52858963E-01	-0.49555320E-01	-0.39644288E-01	-0.23125853E-01	0.
	-0.48089026E-01	-0.51294913E-01	-0.48089026E-01	-0.38471256E-01	-0.22441586E-01	0.
	-0.46625824E-01	-0.49734165E-01	-0.46625824E-01	-0.37300695E-01	-0.21759760E-01	0.
	-0.45165636E-01	-0.48176631E-01	-0.45165636E-01	-0.36132544E-01	-0.21077339E-01	0.
	-0.43708359E-01	-0.46622209E-01	-0.43708359E-01	-0.34966719E-01	-0.20397272E-01	0.
	-0.42253085E-01	-0.45070775E-01	-0.42253085E-01	-0.33803136E-01	-0.19718513E-01	0.
	-0.40802070E-01	-0.43522176E-01	-0.40802070E-01	-0.32641680E-01	-0.19040994E-01	0.
	-0.39352730E-01	-0.41976219E-01	-0.39352730E-01	-0.31482203E-01	-0.18364630E-01	0.
	-0.37905699E-01	-0.40432724E-01	-0.37905699E-01	-0.30324575E-01	-0.17689344E-01	0.
	-0.36460824E-01	-0.38891528E-01	-0.36460824E-01	-0.29169673E-01	-0.17015067E-01	0.
	-0.35017957E-01	-0.37352473E-01	-0.35017957E-01	-0.28014378E-01	-0.16341726E-01	0.
	-0.33576968E-01	-0.35815421E-01	-0.33576968E-01	-0.26861584E-01	-0.15669261E-01	0.
	-0.32137783E-01	-0.34280293E-01	-0.32137783E-01	-0.25710233E-01	-0.14997539E-01	0.
	-0.30700307E-01	-0.32746988E-01	-0.30700307E-01	-0.24560250E-01	-0.14326815E-01	0.
	-0.29264434E-01	-0.31215392E-01	-0.29264434E-01	-0.23411549E-01	-0.13656738E-01	0.
	-0.27830150E-01	-0.29685492E-01	-0.27830150E-01	-0.22264120E-01	-0.12987404E-01	0.
	-0.26397418E-01	-0.28157245E-01	-0.26397418E-01	-0.21117936E-01	-0.12318797E-01	0.
	-0.24966154E-01	-0.26630561E-01	-0.24966154E-01	-0.19972925E-01	-0.11650474E-01	0.
	-0.23536307E-01	-0.25105394E-01	-0.23536307E-01	-0.18829048E-01	-0.10983612E-01	0.
	-0.22107886E-01	-0.23581743E-01	-0.22107886E-01	-0.17686310E-01	-0.10317014E-01	0.
	-0.20680898E-01	-0.22059624E-01	-0.20680898E-01	-0.16544719E-01	-0.96510866E-02	0.
	-0.19255328E-01	-0.20539017E-01	-0.19255328E-01	-0.15404263E-01	-0.89858206E-02	0.
	-0.17831162E-01	-0.19019906E-01	-0.17831162E-01	-0.14264930E-01	-0.83212099E-02	0.
	-0.16408396E-01	-0.17502289E-01	-0.16408396E-01	-0.13126717E-01	-0.76572518E-02	0.
	-0.14987014E-01	-0.15986148E-01	-0.14987014E-01	-0.11989611E-01	-0.69939406E-02	0.
	-0.13567010E-01	-0.14471477E-01	-0.13567010E-01	-0.10853608E-01	-0.63312714E-02	0.
	-0.12148383E-01	-0.12958276E-01	-0.12148383E-01	-0.97187060E-02	-0.56692447E-02	0.
	-0.10731136E-01	-0.11446545E-01	-0.10731136E-01	-0.85849085E-02	-0.50078627E-02	0.
	-0.93152600E-02	-0.99362777E-02	-0.93152600E-02	-0.74522085E-02	-0.43471216E-02	0.
	-0.79007465E-02	-0.84274638E-02	-0.79007465E-02	-0.63205979E-02	-0.36870158E-02	0.
	-0.64875924E-02	-0.69200985E-02	-0.64875924E-02	-0.51900745E-02	-0.30275438E-02	0.
	-0.50757927E-02	-0.54141794E-02	-0.50757927E-02	-0.40606344E-02	-0.23687032E-02	0.
	-0.36653633E-02	-0.39097209E-02	-0.36653633E-02	-0.29322902E-02	-0.17105029E-02	0.
	-0.22562941E-02	-0.24067135E-02	-0.22562941E-02	-0.18050358E-02	-0.10529377E-02	0.
	-0.84857836E-03	-0.90515018E-03	-0.84857836E-03	-0.67886223E-03	-0.39600253E-03	0.
	0.55779205E-03	0.59497797E-03	0.55779205E-03	0.44623374E-03	0.26030283E-03	0.
	0.19628300E-02	0.20936853E-02	0.19628300E-02	0.15702637E-02	0.91598744E-03	0.
	0.33665368E-02	0.35909724E-02	0.33665368E-02	0.26932302E-02	0.15710516E-02	0.
	0.47689098E-02	0.50868377E-02	0.47689098E-02	0.38151276E-02	0.22254904E-02	0.
	0.61699547E-02	0.65812863E-02	0.61699547E-02	0.49359626E-02	0.28793108E-02	0.
	0.75696887E-02	0.80743355E-02	0.75696887E-02	0.60557506E-02	0.35325208E-02	0.
	0.89681058E-02	0.95659795E-02	0.89681058E-02	0.71744844E-02	0.41851154E-02	0.
	0.10365207E-01	0.11056221E-01	0.10365207E-01	0.82921661E-02	0.48370969E-02	0.
	0.11761000E-01	0.12545060E-01	0.11761000E-01	0.94098005E-02	0.54884667E-02	0.
	0.13155483E-01	0.14032514E-01	0.13155483E-01	0.10524386E-01	0.61392260E-02	0.
	0.14548653E-01	0.15518563E-01	0.14548653E-01	0.11638924E-01	0.67893736E-02	0.
	0.15940522E-01	0.17003224E-01	0.15940522E-01	0.12752418E-01	0.74389111E-02	0.
	0.17331091E-01	0.18486497E-01	0.17331091E-01	0.13864873E-01	0.80878425E-02	0.
	0.18720360E-01	0.19968385E-01	0.18720360E-01	0.14976288E-01	0.87361674E-02	0.
	0.20108347E-01	0.21449904E-01	0.20108347E-01	0.16086677E-01	0.93838950E-02	0.
	0.21495059E-01	0.229[illegible]3E-01	0.21495059E-01	0.17196047E-01	0.10031028E-01	0.
	0.22880509E-01	0.244[illegible]7E-01	0.22880509E-01	0.18304408E-01	0.10677572E-01	0.
	0.24264712E-01	0.25[illegible]9E-01	0.24264712E-01	0.19411770E-01	0.11323533E-01	0.
	0.25647675E-01	0.27357519E-01	0.25647675E-01	0.20518141E-01	0.11968916E-01	0.
	0.27029392E-01	0.28831350E-01	0.27029392E-01	0.21623516E-01	0.12613719E-01	0.
	0.28409880E-01	0.30303869E-01	0.28409880E-01	0.22727906E-01	0.13257946E-01	0.
	0.29789177E-01	0.31775118E-01	0.29789177E-01	0.23831344E-01	0.13901618E-01	0.
	0.31167372E-01	0.33245191E-01	0.31167372E-01	0.24933902E-01	0.14544779E-01	0.
	0.32544500E-01	0.34714126E-01	0.32544500E-01	0.26035607E-01	0.15187441E-01	0.
	0.33920614E-01	0.36181980E-01	0.33920614E-01	0.27136499E-01	0.15829628E-01	0.
	0.35295805E-01	0.37648845E-01	0.35295805E-01	0.28236654E-01	0.16471387E-01	0.
	0.36670161E-01	0.39114821E-01	0.36670161E-01	0.29336143E-01	0.17112758E-01	0.
	0.38043844E-01	0.40580080E-01	0.38043844E-01	0.30435092E-01	0.17753913E-01	0.
	0.39416982E-01	0.42044755E-01	0.39416982E-01	0.31533605E-01	0.18394615E-01	0.
	0.40789706E-01	0.43508988E-01	0.40789706E-01	0.32631787E-01	0.19035223E-01	0.
	0.42162210E-01	0.44972989E-01	0.42162210E-01	0.33729795E-01	0.19675729E-01	0.
	0.43534682E-01	0.46436953E-01	0.43534682E-01	0.34827776E-01	0.20316221E-01	0.
	0.44907254E-01	0.47901025E-01	0.44907254E-01	0.35925838E-01	0.20956758E-01	0.
	0.46280066E-01	0.49365356E-01	0.46280066E-01	0.37024087E-01	0.21597404E-01	0.
	0.47653279E-01	0.50830116E-01	0.47653279E-01	0.38122658E-01	0.22238237E-01	0.
	0.49026918E-01	0.52295333E-01	0.49026918E-01	0.39221560E-01	0.22879268E-01	0.
	0.50400855E-01	0.53760873E-01	0.50400855E-01	0.40320713E-01	0.23520432E-01	0.
	0.51774077E-01	0.55226506E-01	0.51774077E-01	0.41419923E-01	0.24161635E-01	0.
	0.53148619E-01	0.56691846E-01	0.53148619E-01	0.42518906E-01	0.24802701E-01	0.
	0.54521587E-01	0.58156366E-01	0.54521587E-01	0.43617266E-01	0.25443403E-01	0.
	0.55893126E-01	0.59619366E-01	0.55893126E-01	0.44714478E-01	0.26083432E-01	0.
	0.57262438E-01	0.61079994E-01	0.57262438E-01	0.45809904E-01	0.26722418E-01	0.
	0.58628620E-01	0.62537288E-01	0.58628620E-01	0.46902825E-01	0.27359439E-01	0.
	0.59990775E-01	0.63990291E-01	0.59990775E-01	0.47992515E-01	0.27995569E-01	0.
	0.61348274E-01	0.65438339E-01	0.61348274E-01	0.49078459E-01	0.28628996E-01	0.
	0.62701926E-01	0.66882335E-01	0.62701926E-01	0.50161217E-01	0.29260471E-01	0.
Row d	0.64059105E-01	0.68330501E-01	0.64059105E-01	0.51245975E-01	0.29892495E-01	0.
Row c	0.65457626E-01	0.69826590E-01	0.65457626E-01	0.52356053E-01	0.30533299E-01	0.
Row b	0.67080878E-01	0.71593422E-01	0.67080878E-01	0.53579802E-01	0.31179389E-01	0.
Row a	0.69549051E-01	0.74408997E-01	0.69549051E-01	0.55161100E-01	0.31525523E-01	0.
	0.68996896E-01	0.73755702E-01	0.68996894E-01	0.54857050E-01	0.31477470E-01	0.
Outside Edge	0.	0.39838655E-01	0.42142928E-01	0.39400101E-01	0.62748731E-01	0.

M ARRAY

	Column B'	Column A	Column B	Column C	Column D	Column E
Inside Edge	0.	0.	0.77650482E-01	[illegible]E-00	0.2257159[illegible]E-00	0.
Row a'	-0.44561546E-02	0.	0.44561546E-02	0.7063496[illegible]E-02	0.8514465[illegible]	[illegible]
Row b'	0.38795925E-03	0.	-0.38795925E-03	-0.69506700E-03	-0.82696050E-03	-0.99999999E-03
Row c'	0.39510076E-03	0.	-0.39510076E-03	-0.72181053E-03	-0.92927173E-03	-0.99999999E-03
	0.36709151E-03	0.	-0.36709151E-03	-0.68600374E-03	-0.91104605E-03	-0.99999999E-03
	0.36387092E-03	0.	-0.36387092E-03	-0.68208507E-03	-0.90920097E-03	-0.99999999E-03
	0.36364640E-03	0.	-0.36364640E-03	-0.68182776E-03	-0.90909364E-03	-0.99999999E-03
	0.36363532E-03	0.	-0.36363532E-03	-0.68181673E-03	-0.90908969E-03	-0.99999999E-03
	0.36363678E-03	0.	-0.36363678E-03	-0.68181822E-03	-0.90909073E-03	-0.99999999E-03
	0.36363779E-03	0.	-0.36363779E-03	-0.68181441E-03	-0.90909035E-03	-0.99999999E-03
	0.36363515E-03	0.	-0.36363515E-03	-0.68181561E-03	-0.90908911E-03	-0.99999999E-03
	0.36363824E-03	0.	-0.36363824E-03	-0.68182015E-03	-0.90909121E-03	-0.99999999E-03
	0.36363690E-03	0.	-0.36363690E-03	-0.68181856E-03	-0.90909052E-03	-0.99999999E-03
	0.36363646E-03	0.	-0.36363646E-03	-0.68181900E-03	-0.90909066E-03	-0.99999999E-03
	0.36363718E-03	0.	-0.36363718E-03	-0.68181794E-03	-0.90909110E-03	-0.99999999E-03
	0.36363465E-03	0.	-0.36363465E-03	-0.68181944E-03	-0.90909097E-03	-0.99999999E-03
	0.36363690E-03	0.	-0.36363690E-03	-0.68181988E-03	-0.90909022E-03	-0.99999999E-03
	0.36363500E-03	0.	-0.36363500E-03	-0.68181720E-03	-0.90908901E-03	-0.99999999E-03
	0.36363518E-03	0.	-0.36363518E-03	-0.68181988E-03	-0.90909190E-03	-0.99999999E-03
	0.36363602E-03	0.	-0.36363602E-03	-0.68181748E-03	-0.90909089E-03	-0.99999999E-03
	0.36364060E-03	0.	-0.36364060E-03	-0.68182177E-03	-0.90909290E-03	-0.99999999E-03
	0.36363818E-03	0.	-0.36363818E-03	-0.68181935E-03	-0.90909001E-03	-0.99999999E-03
	0.36363576E-03	0.	-0.36363576E-03	-0.68181691E-03	-0.90909047E-03	-0.99999999E-03
	0.36363231E-03	0.	-0.36363231E-03	-0.68181453E-03	-0.90908742E-03	-0.99999999E-03
	0.36363413E-03	0.	-0.36363413E-03	-0.68181606E-03	-0.90909006E-03	-0.99999999E-03
	0.36363038E-03	0.	-0.36363038E-03	-0.68181179E-03	-0.90908637E-03	-0.99999999E-03
	0.36363242E-03	0.	-0.36363242E-03	-0.68181368E-03	-0.90908544E-03	-0.99999999E-03
	0.36363734E-03	0.	-0.36363734E-03	-0.68181939E-03	-0.90909285E-03	-0.99999999E-03
	0.36363440E-03	0.	-0.36363440E-03	-0.68181644E-03	-0.90909044E-03	-0.99999999E-03
	0.36363376E-03	0.	-0.36363376E-03	-0.68181850E-03	-0.90909126E-03	-0.99999999E-03
	0.36363652E-03	0.	-0.36363652E-03	-0.68181518E-03	-0.90908975E-03	-0.99999999E-03
	0.36363440E-03	0.	-0.36363440E-03	-0.68181642E-03	-0.90909030E-03	-0.99999999E-03
	0.36363434E-03	0.	-0.36363434E-03	-0.68181649E-03	-0.90909033E-03	-0.99999999E-03
	0.36363871E-03	0.	-0.36363871E-03	-0.68181986E-03	-0.90909190E-03	-0.99999999E-03
	0.36363710E-03	0.	-0.36363710E-03	-0.68181752E-03	-0.90909083E-03	-0.99999999E-03
	0.36364010E-03	0.	-0.36364010E-03	-0.68181960E-03	-0.90909142E-03	-0.99999999E-03
	0.36363654E-03	0.	-0.36363654E-03	-0.6818184[illegible]E-03	-0.90909128E-03	-0.99999999E-03
	0.36363354E-03	0.	-0.36363354E-03	-0.68181574E-03	-0.90908997E-03	-0.99999999E-03
	0.36363560E-03	0.	-0.36363560E-03	-0.68181586E-03	-0.90908998E-03	-0.99999999E-03
	0.36363079E-03	0.	-0.36363079E-03	-0.68181381E-03	-0.90908730E-03	-0.99999999E-03
	0.36363219E-03	0.	-0.36363219E-03	-0.68181511E-03	-0.90908790E-03	-0.99999999E-03
	0.36363214E-03	0.	-0.36363214E-03	-0.68181771E-03	-0.90908909E-03	-0.99999999E-03
	0.36363203E-03	0.	-0.36363203E-03	-0.68181337E-03	-0.90908705E-03	-0.99999999E-03
	0.36363453E-03	0.	-0.36363453E-03	-0.68181571E-03	-0.90908988E-03	-0.99999999E-03
	0.36363246E-03	0.	-0.36363246E-03	-0.68181188E-03	-0.90908812E-03	-0.99999999E-03
	0.36363441E-03	0.	-0.36363441E-03	-0.68181551E-03	-0.90908977E-03	-0.99999999E-03
	0.36363474E-03	0.	-0.36363474E-03	-0.68181596E-03	-0.90909012E-03	-0.99999999E-03
	0.36363561E-03	0.	-0.36363561E-03	-0.68181756E-03	-0.90909082E-03	-0.99999999E-03
	0.36363521E-03	0.	-0.36363521E-03	-0.68181646E-03	-0.90909032E-03	-0.99999999E-03
	0.36363336E-03	0.	-0.36363336E-03	-0.68181727E-03	-0.90908701E-03	-0.99999999E-03
	0.36363570E-03	0.	-0.36363570E-03	-0.68181784E-03	-0.90909097E-03	-0.99999999E-03
	0.36363541E-03	0.	-0.36363541E-03	-0.68181568E-03	-0.90908990E-03	-0.99999999E-03
	0.36363478E-03	0.	-0.36363478E-03	-0.68181514E-03	-0.90909141E-03	-0.99999999E-03
	0.36363307E-03	0.	-0.36363307E-03	-0.68181505E-03	-0.90908964E-03	-0.99999999E-03
	0.36362854E-03	0.	-0.36362854E-03	-0.68181256E-03	-0.90908671E-03	-0.99999999E-03
	0.36363185E-03	0.	-0.36363185E-03	-0.68181491E-03	-0.90908774E-03	-0.99999999E-03
	0.36363591E-03	0.	-0.36363591E-03	-0.68181714E-03	-0.90909063E-03	-0.99999999E-03
	0.36363613E-03	0.	-0.36363613E-03	-0.68181814E-03	-0.90908942E-03	-0.99999999E-03
	0.36363576E-03	0.	-0.36363576E-03	-0.68181799E-03	-0.90908928E-03	-0.99999999E-03
	0.36363827E-03	0.	-0.36363827E-03	-0.68181679E-03	-0.90909042E-03	-0.99999999E-03
	0.36363511E-03	0.	-0.36363511E-03	-0.68181808E-03	-0.90908936E-03	-0.99999999E-03
	0.36363679E-03	0.	-0.36363679E-03	-0.68181793E-03	-0.90909100E-03	-0.99999999E-03
	0.36363237E-03	0.	-0.36363237E-03	-0.68181189E-03	-0.90908918E-03	-0.99999999E-03
	0.36363324E-03	0.	-0.36363324E-03	-0.68181521E-03	-0.90908963E-03	-0.99999999E-03
	0.36363069E-03	0.	-0.36363069E-03	-0.68181378E-03	-0.90908723E-03	-0.99999999E-03
	0.36363566E-03	0.	-0.36363566E-03	-0.68181424E-03	-0.90908924E-03	-0.99999999E-03
	0.36363561E-03	0.	-0.36363561E-03	-0.68181917E-03	-0.90908967E-03	-0.99999999E-03
	0.36363430E-03	0.	-0.36363430E-03	-0.68181226E-03	-0.90908835E-03	-0.99999999E-03
	0.36363312E-03	0.	-0.36363312E-03	-0.68181412E-03	-0.90908916E-03	-0.99999999E-03
	0.36363371E-03	0.	-0.36363371E-03	-0.68181574E-03	-0.90908822E-03	-0.99999999E-03
	0.36363330E-03	0.	-0.36363330E-03	-0.68181627E-03	-0.90909019E-03	-0.99999999E-03
	0.36363442E-03	0.	-0.36363442E-03	-0.68181655E-03	-0.90909037E-03	-0.99999999E-03
	0.36363626E-03	0.	-0.36363626E-03	-0.68181735E-03	-0.90909255E-03	-0.99999999E-03
	0.36363217E-03	0.	-0.36363217E-03	-0.68181780E-03	-0.90908933E-03	-0.99999999E-03
	0.36363415E-03	0.	-0.36363415E-03	-0.68181890E-03	-0.90908969E-03	-0.99999999E-03
	0.36363224E-03	0.	-0.36363224E-03	-0.68181358E-03	-0.90908538E-03	-0.99999999E-03
	0.36363333E-03	0.	-0.36363333E-03	-0.68181655E-03	-0.90909209E-03	-0.99999999E-03
	0.36363646E-03	0.	-0.36363646E-03	-0.68182090E-03	-0.90909242E-03	-0.99999999E-03
	0.36363236E-03	0.	-0.36363236E-03	-0.68181475E-03	-0.90909136E-03	-0.99999999E-03
	0.36363663E-03	0.	-0.36363663E-03	-0.68181773E-03	-0.90908921E-03	-0.99999999E-03
	0.36363447E-03	0.	-0.36363447E-03	-0.68181670E-03	-0.90909041E-03	-0.99999999E-03
	0.36363589E-03	0.	-0.36363589E-03	-0.68181542E-03	-0.90908812E-03	-0.99999999E-03
	0.36363701E-03	0.	-0.36363701E-03	-0.68181807E-03	-0.90909102E-03	-0.99999999E-03
	0.36363422E-03	0.	-0.36363422E-03	-0.68181635E-03	-0.90909031E-03	-0.99999999E-03
	0.36363432E-03	0.	-0.36363432E-03	-0.68181641E-03	-0.90909025E-03	-0.99999999E-03
	0.36363663E-03	0.	-0.36363663E-03	-0.68181745E-03	-0.90909163E-03	-0.99999999E-03
	0.36363576E-03	0.	-0.36363576E-03	-0.68181645E-03	-0.90909120E-03	-0.99999999E-03
	0.36363474E-03	0.	-0.36363474E-03	-0.68181510E-03	-0.90908973E-03	-0.99999999E-03
	0.36363482E-03	0.	-0.36363482E-03	-0.68181597E-03	-0.90908922E-03	-0.99999999E-03
	0.36363475E-03	0.	-0.36363475E-03	-0.68181584E-03	-0.90908913E-03	-0.99999999E-03
	0.36363374E-03	0.	-0.36363374E-03	-0.68181585E-03	-0.90909001E-03	-0.99999999E-03
	0.36363536E-03	0.	-0.36363536E-03	-0.68181788E-03	-0.90909190E-03	-0.99999999E-03
	0.36363537E-03	0.	-0.36363537E-03	-0.68181740E-03	-0.90908998E-03	-0.99999999E-03
	0.36363456E-03	0.	-0.36363456E-03	-0.68181619E-03	-0.90909021E-03	-0.99999999E-03
	0.36363466E-03	0.	-0.36363466E-03	-0.68181725E-03	-0.90909030E-03	-0.99999999E-03
	0.36363684E-03	0.	-0.36363684E-03	-0.68181846E-03	-0.90909051E-03	-0.99999999E-03
	0.36364713E-03	0.	-0.36364713E-03	-0.68182495E-03	-0.90909409E-03	-0.99999999E-03
	0.36386415E-03	0.	-0.36386415E-03	-0.68207657E-03	-0.90919720E-03	-0.99999999E-03
Row c	0.36699688E-03	0.	-0.36699688E-03	-0.68588906E-03	-0.91103103E-03	-0.99999999E-03
Row b	0.39423104E-03	0.	-0.39423104E-03	-0.72070380E-03	-0.92871379E-03	-0.99999999E-03
Row a	0.38776365E-03	0.	-0.38776365E-03	-0.69541036E-03	-0.82993055E-03	-0.99999999E-03
	-0.43044931E-02	0.	0.43044931E-02	0.68201099E-02	0.82197364E-02	-0.99999999E-03
Outside Edge	0.	0.	0.75201905E-01	0.14938126E-00	0.21860147E-00	0.

Plane of Symmetry, Z = 0

P ARRAY

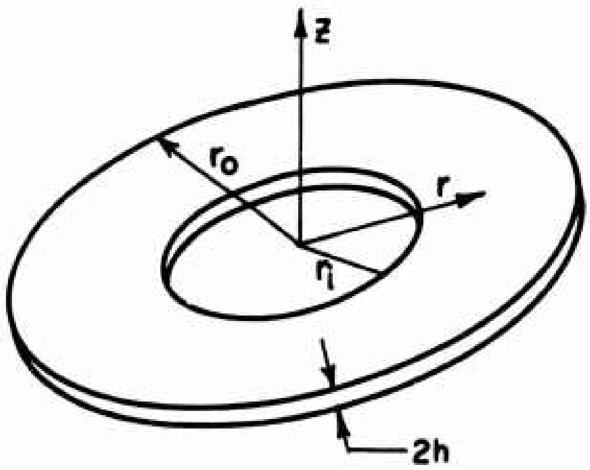
	Column B'	Column A	Column B	Column C	Column D	Column E
Inside Edge	0.	0.	0.	0.	0.	0.
Row a'	0.24113346E-01	0.35497231E-01	0.24113346E-01	0.16128328E-01	-0.32320506E-01	0.
Row b'	-0.14307791E-00	-0.14222154E-00	-0.14307791E-00	-0.14520450E-00	-0.14134265E-00	-0.14957246E-00
Row c'	-0.29248539E-00	-0.29233523E-00	-0.29248539E-00	-0.29289767E-00	-0.29346844E-00	-0.29404452E-00
	-0.43751300E-00	-0.43742056E-00	-0.43751300E-00	-0.43779028E-00	-0.43824438E-00	-0.43886416E-00
	-0.57938277E 00	-0.57929394E 00	-0.57938277E 00	-0.57965339E 00	-0.58010542E 00	-0.58073473E 00
	-0.71813046E 00	-0.71804214E 00	-0.71813046E 00	-0.71840058E 00	-0.71885251E 00	-0.71948624E 00
	-0.85374466E 00	-0.85365670E 00	-0.85374466E 00	-0.85401444E 00	-0.85446604E 00	-0.85509944E 00
	-0.98621630E 00	-0.98612861E 00	-0.98621630E 00	-0.98648579E 00	-0.98693711E 00	-0.98757025E 00
	-0.11155390E 01	-0.11154515E 01	-0.11155390E 01	-0.11158093E 01	-0.11162593E 01	-0.11168922E 01
	-0.12417090E 01	-0.12416217E 01	-0.12417090E 01	-0.12419781E 01	-0.12424291E 01	-0.12430619E 01
	-0.13667246E 01	-0.13666374E 01	-0.13667246E 01	-0.13669936E 01	-0.13674445E 01	-0.13680771E 01
	-0.14845861E 01	-0.14844999E 01	-0.14845861E 01	-0.14848550E 01	-0.14853059E 01	-0.14859385E 01
	-0.16012957E 01	-0.16012085E 01	-0.16012957E 01	-0.16015647E 01	-0.16020156E 01	-0.16026482E 01
	-0.17148571E 01	-0.17147698E 01	-0.17148571E 01	-0.17151261E 01	-0.17155770E 01	-0.17162047E 01
	-0.18252750E 01	-0.18251877E 01	-0.18252750E 01	-0.18255442E 01	-0.18259951E 01	-0.18266279E 01
	-0.19325551E 01	-0.19324677E 01	-0.19325551E 01	-0.19328243E 01	-0.19332754E 01	-0.19339092E 01
	-0.20367039E 01	-0.20366164E 01	-0.20367039E 01	-0.20369732E 01	-0.20374244E 01	-0.20380573E 01
	-0.21377280E 01	-0.21376405E 01	-0.21377280E 01	-0.21379974E 01	-0.21384487E 01	-0.21390817E 01
	-0.22356342E 01	-0.22355465E 01	-0.22356342E 01	-0.22359037E 01	-0.22363549E 01	-0.22369881E 01
	-0.23304284E 01	-0.23303407E 01	-0.23304284E 01	-0.23306980E 01	-0.23311494E 01	-0.23317826E 01
	-0.24221169E 01	-0.24220291E 01	-0.24221169E 01	-0.24223866E 01	-0.24228381E 01	-0.24234714E 01
	-0.25107056E 01	-0.25106176E 01	-0.25107056E 01	-0.25109753E 01	-0.25114268E 01	-0.25120602E 01
	-0.25961997E 01	-0.25961118E 01	-0.25961997E 01	-0.25964696E 01	-0.25969212E 01	-0.25975546E 01
	-0.26786046E 01	-0.26785167E 01	-0.26786046E 01	-0.26788745E 01	-0.26793262E 01	-0.26799596E 01
	-0.27579250E 01	-0.27578370E 01	-0.27579250E 01	-0.27581949E 01	-0.27586465E 01	-0.27592801E 01
	-0.28341652E 01	-0.28340771E 01	-0.28341652E 01	-0.28344351E 01	-0.28348968E 01	-0.28355204E 01
	-0.29073294E 01	-0.29072412E 01	-0.29073294E 01	-0.29075993E 01	-0.29080510E 01	-0.29086945E 01
	-0.29774213E 01	-0.29773331E 01	-0.29774213E 01	-0.29776912E 01	-0.29781430E 01	-0.29787766E 01
	-0.30444450E 01	-0.30443569E 01	-0.30444450E 01	-0.30447149E 01	-0.30451667E 01	-0.30458003E 01
	-0.31084038E 01	-0.31083156E 01	-0.31084038E 01	-0.31086738E 01	-0.31091256E 01	-0.31097592E 01
	-0.31693010E 01	-0.31692129E 01	-0.31693010E 01	-0.31695709E 01	-0.31700227E 01	-0.31706564E 01
	-0.32271402E 01	-0.32270520E 01	-0.32271402E 01	-0.32274101E 01	-0.32278620E 01	-0.32284956E 01
	-0.32819245E 01	-0.32818363E 01	-0.32819245E 01	-0.32821945E 01	-0.32826463E 01	-0.32832799E 01
	-0.33336570E 01	-0.33335689E 01	-0.33336570E 01	-0.33339270E 01	-0.33343788E 01	-0.33350125E 01
	-0.33823407E 01	-0.33822525E 01	-0.33823407E 01	-0.33826107E 01	-0.33830625E 01	-0.33836962E 01
	-0.34279786E 01	-0.34278904E 01	-0.34279786E 01	-0.34282485E 01	-0.34287003E 01	-0.34293340E 01
	-0.34705737E 01	-0.34704856E 01	-0.34705737E 01	-0.34708437E 01	-0.34712955E 01	-0.34719292E 01
	-0.35101292E 01	-0.35100410E 01	-0.35101292E 01	-0.35103992E 01	-0.35108510E 01	-0.35114846E 01
	-0.35466480E 01	-0.35465597E 01	-0.35466480E 01	-0.35469180E 01	-0.35473698E 01	-0.35480034E 01
	-0.35801329E 01	-0.35800447E 01	-0.35801329E 01	-0.35804030E 01	-0.35808548E 01	-0.35814884E 01
	-0.36105870E 01	-0.36104989E 01	-0.36105870E 01	-0.36108571E 01	-0.36113089E 01	-0.36119425E 01
	-0.36380134E 01	-0.36379252E 01	-0.36380134E 01	-0.36382834E 01	-0.36387352E 01	-0.36393689E 01
	-0.36624147E 01	-0.36623266E 01	-0.36624147E 01	-0.36626847E 01	-0.36631365E 01	-0.36637702E 01
	-0.36837941E 01	-0.36837060E 01	-0.36837941E 01	-0.36840641E 01	-0.36845159E 01	-0.36851496E 01
	-0.37021544E 01	-0.37020662E 01	-0.37021544E 01	-0.37024244E 01	-0.37028762E 01	-0.37035099E 01
	-0.37174986E 01	-0.37174103E 01	-0.37174986E 01	-0.37177685E 01	-0.37182204E 01	-0.37188540E 01
	-0.37298293E 01	-0.37297412E 01	-0.37298293E 01	-0.37300994E 01	-0.37305512E 01	-0.37311848E 01
	-0.37391498E 01	-0.37390616E 01	-0.37391498E 01	-0.37394197E 01	-0.37398716E 01	-0.37405052E 01
	-0.37454627E 01	-0.37453746E 01	-0.37454627E 01	-0.37457328E 01	-0.37461846E 01	-0.37468182E 01
	-0.37487712E 01	-0.37486830E 01	-0.37487712E 01	-0.37490412E 01	-0.37494930E 01	-0.37501266E 01
	-0.37490779E 01	-0.37489898E 01	-0.37490779E 01	-0.37493479E 01	-0.37497997E 01	-0.37504334E 01
	-0.37463858E 01	-0.37462977E 01	-0.37463858E 01	-0.37466558E 01	-0.37471077E 01	-0.37477414E 01
	-0.37406977E 01	-0.37406096E 01	-0.37406977E 01	-0.37409677E 01	-0.37414195E 01	-0.37420531E 01
	-0.37320165E 01	-0.37319283E 01	-0.37320165E 01	-0.37322865E 01	-0.37327383E 01	-0.37333719E 01
	-0.37203450E 01	-0.37202569E 01	-0.37203450E 01	-0.37206150E 01	-0.37210668E 01	-0.37217004E 01
	-0.37056860E 01	-0.37055977E 01	-0.37056860E 01	-0.37059559E 01	-0.37064077E 01	-0.37070414E 01
	-0.36880422E 01	-0.36879540E 01	-0.36880422E 01	-0.36883122E 01	-0.36887640E 01	-0.36893976E 01
	-0.36674166E 01	-0.36673284E 01	-0.36674166E 01	-0.36676866E 01	-0.36681384E 01	-0.36687720E 01
	-0.36438119E 01	-0.36437237E 01	-0.36438119E 01	-0.36440819E 01	-0.36445337E 01	-0.36451673E 01
	-0.36172308E 01	-0.36171427E 01	-0.36172308E 01	-0.36175009E 01	-0.36179527E 01	-0.36185863E 01
	-0.35876763E 01	-0.35875881E 01	-0.35876763E 01	-0.35879463E 01	-0.35883981E 01	-0.35890318E 01
	-0.35551511E 01	-0.35550629E 01	-0.35551511E 01	-0.35554211E 01	-0.35558729E 01	-0.35565066E 01
	-0.35196579E 01	-0.35195698E 01	-0.35196579E 01	-0.35199279E 01	-0.35203797E 01	-0.35210133E 01
	-0.34811997E 01	-0.34811115E 01	-0.34811997E 01	-0.34814697E 01	-0.34819215E 01	-0.34825551E 01
	-0.34397790E 01	-0.34396908E 01	-0.34397790E 01	-0.34400489E 01	-0.34405008E 01	-0.34411344E 01
	-0.33953986E 01	-0.33953104E 01	-0.33953986E 01	-0.33956686E 01	-0.33961204E 01	-0.33967540E 01
	-0.33480612E 01	-0.33479731E 01	-0.33480612E 01	-0.33483312E 01	-0.33487830E 01	-0.33494167E 01
	-0.32977697E 01	-0.32976815E 01	-0.32977697E 01	-0.32980397E 01	-0.32984915E 01	-0.32991251E 01
	-0.32445263E 01	-0.32444382E 01	-0.32445263E 01	-0.32447963E 01	-0.32452481E 01	-0.32458818E 01
	-0.31883341E 01	0.31882460E 01	-0.31883341E 01	-0.31886040E 01	-0.31890558E 01	-0.31896895E 01
	-0.31291954E 01	-0.31291073E 01	-0.31291954E 01	-0.31294654E 01	-0.31299172E 01	-0.31305508E 01
	-0.30671131E 01	-0.30670249E 01	-0.30671131E 01	-0.30673830E 01	-0.30678349E 01	-0.30684485E 01
	-0.30020895E 01	-0.30020013E 01	-0.30020895E 01	-0.30023595E 01	-0.30028113E 01	-0.30034449E 01
	-0.29341268E 01	-0.29340386E 01	-0.29341268E 01	-0.29343967E 01	-0.29348485E 01	-0.29354821E 01
	-0.28632274E 01	-0.28631393E 01	-0.28632274E 01	-0.28634974E 01	-0.28639491E 01	-0.28645827E 01
	-0.27893932E 01	-0.27893052E 01	-0.27893932E 01	-0.27896632E 01	-0.27901150E 01	-0.27907485E 01
	-0.27126263E 01	-0.27125382E 01	-0.27126263E 01	-0.27128962E 01	-0.27133479E 01	-0.27139813E 01
	-0.26329282E 01	-0.26328402E 01	-0.26329282E 01	-0.26331980E 01	-0.26336497E 01	-0.26342832E 01
	-0.25503001E 01	-0.25502121E 01	-0.25503001E 01	-0.25505699E 01	-0.25510216E 01	-0.25516550E 01
	-0.24647432E 01	-0.24646553E 01	-0.24647432E 01	-0.24650129E 01	-0.24654645E 01	-0.24660978E 01
	-0.23762581E 01	-0.23761702E 01	-0.23762581E 01	-0.23765278E 01	-0.23769792E 01	-0.23776125E 01
	-0.22848450E 01	-0.22847572E 01	-0.22848450E 01	-0.22851146E 01	-0.22855660E 01	-0.22861993E 01
	-0.21905039E 01	-0.21904162E 01	-0.21905039E 01	-0.21907733E 01	-0.21912247E 01	-0.21918579E 01
	-0.20932343E 01	-0.20931467E 01	-0.20932343E 01	-0.20935037E 01	-0.20939549E 01	-0.20945880E 01
	-0.19930357E 01	-0.19929483E 01	-0.19930357E 01	-0.19933051E 01	-0.19937562E 01	-0.19943892E 01
	-0.18899070E 01	-0.18898196E 01	-0.18899070E 01	-0.18901762E 01	-0.18906272E 01	-0.18912601E 01
	-0.17838474E 01	-0.17837600E 01	-0.17838474E 01	-0.17841165E 01	-0.17845674E 01	-0.17852002E 01
	-0.16748562E 01	-0.16747690E 01	-0.16748562E 01	-0.16751253E 01	-0.16755762E 01	-0.16762089E 01
	-0.15629339E 01	-0.15628468E 01	-0.15629339E 01	-0.15632030E 01	-0.15636538E 01	-0.15642864E 01
	-0.14480814E 01	-0.14479943E 01	-0.14480814E 01	-0.14483504E 01	-0.14488012E 01	-0.14494339E 01
	-0.13303011E 01	-0.13302138E 01	-0.13303011E 01	-0.13305701E 01	-0.13310210E 01	-0.13316537E 01
	-0.12095968E 01	-0.12095094E 01	-0.12095968E 01	-0.12098659E 01	-0.12103169E 01	-0.12109496E 01
	-0.10859743E 01	-0.10858868E 01	-0.10859743E 01	-0.10862435E 01	-0.10866947E 01	-0.10873276E 01
	-0.95944150E 00	-0.95935382E 00	-0.95944150E 00	-0.95971101E 00	-0.96016233E 00	-0.96079549E 00
	-0.83000844E 00	-0.82992046E 00	-0.83000844E 00	-0.83027824E 00	-0.83072986E 00	-0.83136331E 00
	-0.69768761E 00	-0.69759926E 00	-0.69768761E 00	-0.69795775E 00	-0.69840971E 00	-0.69904346E 00
	-0.56249426E 00	-0.56240541E 00	-0.56249426E 00	-0.56276490E 00	-0.56321694E 00	-0.56385032E 00
	-0.42444581E-00	-0.42435344E-00	-0.42444581E-00	-0.42472293E-00	-0.42517701E-00	-0.42580111E-00
Row c	-0.28351390E-00	-0.28336544E-00	-0.28351390E-00	-0.28392213E-00	-0.28448943E-00	-0.28507195E-00
Row b	-0.13851930E-00	-0.13768537E-00	-0.13851930E-00	-0.14066918E-00	-0.14365006E-00	-0.14486123E-00
Row a	0.27406122E-01	0.34570341E-01	0.27406122E-01	0.15760998E-01	-0.31223207E-01	0.
Outside Edge	0.	0.	0.	0.	0.	0.

RU = 0.599772E-06 RUU = 0.912181E 00 RW = 0.719447E-07 4VW = 0.907489E 00 RP = 0.746548E-05 RPP = 0.965318E 00

APPENDIX C

APPROXIMATION FOR A COMPRESSIBLE ELASTOMER

Consider a thin disk of rubber bonded between two rigid plates, the rubber being assumed as compressible.



The dimensions of the bearing are denoted by

r_i - inner radius

r_o - outer radius

2h - thickness.

Owing to the difficulties encountered in solving the equations of equilibrium, an approximate solution through variational methods will be derived. In this case, the theorem of minimum potential energy is used. A displacement field, u, w, satisfying the displacement boundary condition, is assumed and put into the potential energy:

$$U = \frac{1}{2} \int_V \sigma_{ij} \epsilon_{ij} \, dV . \tag{C-1}$$

The theorem of minimum potential energy states that

$$\delta U = 0 . \tag{C-2}$$

Thus, by taking the first variation of the expression U, two differential equations and the associated natural boundary conditions are obtained.

Assuming that

$$u = f(r)\left(1 - \frac{z^2}{h^2}\right), \tag{C-3}$$

$$w = \frac{Wo}{h} z + g(r)\; z\; \left(1 - \frac{z^2}{h^2}\right) \tag{C-4}$$

$$\frac{\partial u}{\partial r} = f'(r)\left(1 - \frac{z^2}{h^2}\right)$$

$$\frac{\partial u}{\partial z} = - \frac{2z}{h^2} f(r)$$

$$\frac{\partial w}{\partial r} = g'(r)\; z\; \left(1 - \frac{z^2}{h^2}\right)$$

$$\frac{\partial w}{\partial z} = \frac{Wo}{h} + g(r)\left(1 - 3\frac{z^2}{h^2}\right)$$

$$\frac{u}{\sigma} = \frac{f(r)}{r}\left(1 - \frac{z^2}{h^2}\right).$$

Let

$$\Delta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}. \tag{C-5}$$

Then

$$\sigma_{ij}\; \epsilon_{ij} = W = \lambda\Delta^2 + 2G\, [\epsilon_r\epsilon_r + \epsilon_\theta\epsilon_\theta + \epsilon_z\epsilon_z + 2\epsilon_{rz}\epsilon_{rz}] \tag{C-6}$$

where

$$\Delta = \left(f'(r) + \frac{f}{r}\right)\left(1 - \frac{z^2}{h^2}\right) + \frac{Wo}{h} + g(r)\left(1 - 3\frac{z^2}{h^2}\right),$$

and

$$\Delta^2 = \left[f'(r) + \frac{f}{r}\right]^2 \left(1 - \frac{z^2}{h^2}\right)^2 + \frac{Wo^2}{h^2} + g^2(r)\left(1 - 3\frac{z^2}{h^2}\right)^2$$

$$+ 2\left[f'(r) + \frac{f}{r}\right]\left(1 - \frac{z^2}{h^2}\right)\left(\frac{Wo}{h}\right) + 2\frac{Wo}{h} g(r)\left(1 - 3\frac{z^2}{h^2}\right)$$

$$+ 2\left[f'(r) + \frac{f}{r}\right] g(r)\; \left(1 - \frac{z^2}{h^2}\right)\left(1 - 3\frac{z^2}{h^2}\right).$$

The following are expanded for substitution into equation (C-1):

$$\epsilon_r\epsilon_r = [f'(r)]^2 \left(1 - \frac{z^2}{h^2}\right)^2$$

$$\epsilon_\theta\epsilon_\theta = \frac{f^2}{r^2}\left(1 - \frac{z^2}{h^2}\right)^2$$

$$\epsilon_z\epsilon_z = \left(\frac{Wo}{h}\right)^2 + g^2(r)\left(1 - 3\frac{z^2}{h^2}\right)^2 + 2\left(\frac{Wo}{h}\right) g(r)\left(1 - 3\frac{z^2}{h^2}\right)$$

$$\epsilon_{rz}\epsilon_{rz} = \tfrac{1}{4}\left[- \frac{2z}{h^2} f(r) + g'(r)\, z\left(1 - \frac{z^2}{h^2}\right)\right]^2$$

$$= \tfrac{1}{4}\left[4\frac{z^2}{h^4} f^2(r) + g'^2(r)\, z^2\left(1 - \frac{z^2}{h^2}\right)^2 - 4\frac{z^2}{h^2} f(r)\, g'(r)\left(1- \frac{z^2}{h^2}\right)\right]$$

$$\int_{-h}^{h}\left(1 - \frac{z^2}{h^2}\right)^2 dz = \int_{-h}^{h}\left[1 - 2\frac{z^2}{h^2} + \frac{z^4}{h^4}\right]dz$$

$$= \left[z - \frac{2}{3}\frac{z^3}{h^2} + \frac{1}{5}\frac{z^5}{h^4}\right]_{-h}^{h}$$

$$= 2\left[h - \frac{2}{3}h + \frac{1}{5}h\right] = \frac{16}{15}h$$

$$\int_{-h}^{h}\left(1 - 3\frac{z^2}{h^2}\right)dz = \left(z - \frac{z^3}{h^2}\right)\Big]_{-h}^{h} = 0$$

$$\int_{-h}^{h}\left(1 - \frac{z^2}{h^2}\right)dz = \left[z - \frac{1}{3}\frac{z^3}{h^2}\right]_{-h}^{h} = \frac{4}{3}h$$

$$\int_{-h}^{h}(1 - 3\frac{z^2}{h^2})^2 \, dz = \int_{-h}^{h}(1 - 6\frac{z^2}{h^2} + 9\frac{z^4}{h^4})dz$$

$$= z - 2\frac{z^3}{h^2} + \frac{9}{5}\frac{z^5}{h^4}\Big]_{-h}^{h}$$

$$= 2(h - 2h + \frac{9}{5}) = \frac{8}{5}h$$

$$\int_{-h}^{h}(1 - \frac{z^2}{h^2})(1 - 3\frac{z^2}{h^2})dz = \int_{-h}^{h}(1 - 4\frac{z^2}{h^2} + 3\frac{z^4}{h^4})dz$$

$$= z - \frac{4}{3}\frac{z^3}{h^2} + \frac{3}{5}\frac{z^5}{h^4}\Big]_{-h}^{h}$$

$$= 2(h - \frac{4}{3}h + \frac{3}{5}h) = \frac{8}{15}h$$

$$\int_{-h}^{h} z^2 dz = \frac{z^3}{3}\Big]_{-h}^{h} = \frac{2}{3}h^3$$

$$\int_{-h}^{h} z^2(1 - \frac{z^2}{h^2})^2 = \int_{-h}^{h}(z^2 - z\frac{z^4}{h^4} + \frac{z^6}{h^4})\,dz$$

$$= (\frac{1}{3}z^3 - \frac{2}{5}\frac{z^5}{h^2} + \frac{1}{7}\frac{z^7}{h^4})_{-h}^{h}$$

$$= 2(\frac{h^3}{3} - \frac{2}{5}h^3 + \frac{1}{7}h^3) = \frac{16}{105}h^3$$

$$\int_{-h}^{h} z^2(1 - \frac{z^2}{h^2})d = [\frac{1}{3}z^3 - \frac{z^5}{5}h^2\,]_{-h}^{h}$$

$$= 2(\frac{1}{3}h^3 - \frac{1}{5}h^3) = \frac{4}{15}h^3$$

$$\int \Delta^2\, dz = \frac{16}{15}\, h[f'(r) + \frac{f}{r}]^2 + 2\,\frac{w^2}{h} + \frac{8}{5}\, h\, g^2(r) + \frac{8}{3}\, h(\frac{Wo}{h})[f'(r) + \frac{f}{r}]$$

$$+ \frac{16}{15}\, h\, [f'(r) + \frac{f}{r}]\, g(r)$$

$$\int \epsilon_r \epsilon_r\, dz = \frac{16}{15}\, h\, [f'(r)]^2$$

$$\int \epsilon_\theta \epsilon_\theta\, dz = \frac{16}{15}\, h\, f^2/r^2$$

$$\int \epsilon_z \epsilon_z\, dz = 2h(\frac{Wo}{h})^2 + \frac{8}{5}\, h\, g^2(r)$$

$$\int \epsilon_{rz} \epsilon_{rz}\, dz = \tfrac{1}{4}[\frac{8}{3h}\, f^2(r) + \frac{16}{105}\, h^3\, g'^2(r) - \frac{16}{15}\, h\, f(r)\, g'(r)]\,.$$

Equation (C-1) is equivalent to

$$\int W dV = \{\lambda\Delta^2 + 2G\,(\epsilon_r\epsilon_r + \epsilon_\theta\epsilon_\theta + \epsilon_z\epsilon_z + 2\epsilon_{rz}\epsilon_{rz})\}dV$$

$$= \int \{\lambda\Delta^2 + 2G(\epsilon_r\epsilon_r + \epsilon_\theta\epsilon_\theta + \epsilon_z\epsilon_z + 2\epsilon_{rz}\epsilon_{rz})\}\dot{r}drd\theta dz$$

$$= 2\pi\int \lambda\, \{\frac{16}{15}\, h\, [f'(r) + \frac{f}{r}]^2 + 2\,\frac{Wo^2}{h} + \frac{8}{5}\, h\, g^2(r) + \frac{8}{3}\, h(\frac{Wo}{h})[f'(r) + \frac{f}{r}]$$

$$+ \frac{16}{15}\, h\, [f'(r) + \frac{f}{r}]\, g(r)\}\, rdr$$

$$+ 2\pi\int 2G\, \{\frac{16}{15}\, h\, [f'^2(r) + \frac{f^2}{r^2}] + 2h\, (\frac{Wo^2}{h^2}) + \frac{8}{5}\, h\, g^2(r)$$

$$+ \frac{4}{3h}\, f^2(r) + \frac{8}{105}\, h^3 g'^2(r) - \frac{8}{15}\, h\, f(r)\, g'(r)\}rdr\,; \qquad \text{(C-7)}$$

but

$$\delta \int W dV = 0 \qquad \text{(C-8)}$$

and

$$\int \lambda\{\frac{32}{15} h\,[f'(r) + \frac{f}{r}]\,[\delta f'(r) + \frac{\delta f}{r}] + \frac{16}{5} h\, g(r)\, \delta g(r)$$

$$+ \frac{8}{3} h(\frac{Wo}{h})\,[\delta f'(r) + \frac{\delta f}{r}] + \frac{16}{15} h\,[\delta f'(r) + \frac{\delta f}{r}]g(r)$$

$$+ \frac{16}{15} h\,[f'(r) + \frac{f}{r}]\,\delta g(r)\}rdr$$

$$+\int 2G\,\{\frac{32}{15} h[f'(r)\,\delta f'(r) + \frac{f}{r^2}\,\delta f] + \frac{16}{5} h\, g(r)\,\delta g(r)$$

$$+ \frac{8}{3h} f(r)\,\delta f(r) + \frac{16}{105} h^3\, g'(r)\,\delta g'(r) - \frac{8}{15} h\, f(r)\delta g'(r)$$

$$- \frac{8}{15} h\, g'(r)\,\delta f(r)\}\, rdr = 0. \qquad \text{(C-8a)}$$

Expanding,

$$\int \lambda\,\{\frac{32}{15} h\,[f'(r) + \frac{f}{r}] + \frac{8}{3} h\,(\frac{Wo}{h}) + \frac{16}{15} h\, g(r)\}\,\delta f'(r)rdr$$

$$+\int 2G\,\{\frac{32}{15} h\, f'(r)\}\,\delta f'(r)rdr$$

$$+\int \lambda\,\{\frac{32}{15} h[f'(r) + \frac{f}{r}](\frac{1}{r}) + \frac{8}{3} h(\frac{Wo}{h})\frac{1}{r} + \frac{16}{15} h\,\frac{g(r)}{r}\}\delta f(r)rdr$$

$$+\int 2G\{\frac{32}{15} h\,\frac{f}{r^2} + \frac{8}{3h} f(r) - \frac{8}{15} h\, g'(r)\}\,\delta f(r)\, rdr$$

$$+\int 2G\{\frac{16}{105} h^3\, g'(r) - \frac{8}{15} h\, f(r)\}\,\delta g'(r)rdr$$

$$+\int \lambda\,\{\frac{16}{5} h\, g(r) + \frac{16}{15} h\,[f'(r) + \frac{f}{r}]\}\,\delta g(r)rdr$$

$$+\int 2G\{\frac{16}{5} h\, g(r)\}\,\delta g(r)dr = 0. \qquad \text{(C-8b)}$$

From the first line of equation (C-8b),

$$\int \lambda \{\frac{32}{15} h [f'(r) + \frac{f(r)}{r}] + \frac{8}{3} h (\frac{Wo}{h}) + \frac{16}{15} h g(r)\} \delta f'(r) r dr$$

$$= \int \lambda r\{\frac{32}{15} h[f'(r) + \frac{f(r)}{r}] + \frac{8}{3} h(\frac{Wo}{h}) + \frac{16}{15} h g(r)\}\delta f(r)]_{r_i}^{r_o}$$

$$- \lambda \int \frac{1}{r}\{\frac{32}{15} h[\frac{\partial}{\partial r}[rf'(r)] + f'(r)] + \frac{8}{3} h(\frac{Wo}{h}) + \frac{16}{15} h \frac{\partial}{\partial r}(rg(r)]\}\delta f(r) r dr$$

$$= \lambda r\{\frac{32}{15} h[f'(r) + \frac{f(r)}{r}] + \frac{8}{3h} (\frac{Wo}{h}) + \frac{16}{15} h g(r)\} \delta f(r)]_{r_i}^{r_o}$$

$$- \lambda \int \{\frac{32}{15} h [f''(r) + \frac{2}{r} f'(r)] + \frac{8}{3} h(\frac{Wo}{h}) \frac{1}{r} + \frac{16}{15} h [g'(r)$$

$$+ \frac{g(r)}{r}]\}\delta f(r) r dr.$$

From the second line of equation (C-8b),

$$\int 2G \{\frac{32}{15} h f'(r)\} \delta f'(r) r dr$$

$$= 2Gr\{\frac{32}{15} h f'(r)\} \delta f(r)]_{r_i}^{r_o} - 2G\int \frac{32}{15} h \frac{1}{r} \frac{\partial}{\partial r} [r f'(r)]\delta f(r) dr$$

$$= 2Gr\{\frac{32}{15} h f'(r)\} \delta f(r)]_{r_i}^{r_o} - 2G (\frac{32}{15} h)\int [f''(r) + \frac{f'(r)}{r}]\delta f(r) r dr.$$

From the fifth line of equation (C-8b),

$$\int 2G\{\frac{16}{105} h^3 g'(r) - \frac{8}{15} h f(r)\} \delta g'(r) r dr$$

$$= 2Gr\{\frac{16}{105} h^3 g'(r) - \frac{8}{15} h f(r)\} \delta g(r)]_{r_i}^{r_o}$$

$$- 2G\int \frac{1}{r} \frac{\partial}{\partial r} \{\frac{16}{105} h^3 r g'(r) - \frac{8}{15} h f(r)\} \delta g(r) r dr .$$

Grouping terms in equation (C-8b), we have the differential equations for f and g:

$$\lambda\{\frac{32}{15} h \, [f'' + \frac{1}{r}f' - \frac{f}{r^2}] + \frac{16}{15} hg'\} + 2G\{\frac{32}{15} [f'' + \frac{f'}{r} - \frac{f}{r^2}]$$

$$- \frac{8}{3h} f + \frac{8}{15} h \, g'\} = 0 \qquad \text{(C-9a)}$$

$$\lambda \, \{\frac{16}{15} h \, g + \frac{16}{15} h \, [f' + \frac{1}{r}f]\} + 2G \, \{\frac{16}{5} hg - \frac{16}{105} h^3 \, [g'' + \frac{1}{r} g']$$

$$+ \frac{8}{15} h \, [f' + \frac{1}{r}f]\} = 0. \qquad \text{(C-9b)}$$

After rearrangement, the equations become

$$f'' + \frac{1}{r} f' - \frac{1}{r^2} f - \frac{5}{2h^2} \frac{G}{\lambda+2G} f + \frac{1}{2} \frac{\lambda+G}{\lambda+2G} g' = 0 \qquad \text{(C-10a)}$$

$$g'' + \frac{1}{r} g' - \frac{21}{2h^2} \frac{\lambda+2G}{G} g - \frac{7}{2h^2} \frac{\lambda+G}{G} [f' + \frac{1}{r} f] = 0. \qquad \text{(C-10b)}$$

The appropriate boundary conditions are

$$f' + \frac{\lambda}{\lambda+2G} \{\frac{f}{r} + \frac{1}{2}g + \frac{5}{4} (\frac{W_o}{h})\} = 0 \qquad \text{(C-11a)}$$

$$\frac{2}{7} h^2 \, g' - f = 0 \qquad \text{(C-11b)}$$

at $r = r_i$ and r_o.

For the sake of convenience, these equations are nondimensionalized with respect to h.

Defining

$$\bar{f} = \frac{f}{h}$$

$$\bar{r} = \frac{r}{h}$$

$$\bar{g} = g(\frac{r}{h}), \qquad \text{(C-12)}$$

the set of equations becomes

$$\bar{f}'' + \frac{1}{\bar{r}} \bar{f}' - \frac{1}{\bar{r}^2} \bar{f} - \frac{5}{4} \frac{1-2\nu}{1-\nu} \bar{f} + \frac{1}{4} \frac{1}{1-\nu} \bar{g}' = 0 \tag{C-13a}$$

$$\bar{g}'' + \frac{1}{\bar{r}} \bar{g}' - \frac{21(1-\nu)}{1-2\nu} \bar{g} - \frac{7}{2} \frac{1}{1-2\nu} (\bar{f}' + \frac{1}{\bar{r}} f) = 0, \tag{C-13b}$$

and at r_o and r_i,

$$\bar{f}' + \frac{\nu}{1-\nu} \{\frac{\bar{f}}{\bar{r}} + \frac{1}{2} \bar{g} + \frac{5}{4} (\frac{W_o}{h})\} = 0 \tag{C-14a}$$

$$\frac{2}{7} \bar{g}' - \bar{f} = 0, \tag{C-14b}$$

where prime denotes $d/d\bar{r}$, ν - Poisson's ratio and the following relation is used:

$$\lambda = \frac{2G\nu}{1-2\nu} . \tag{C-15}$$

In the following, the bar will be dropped from quantities b, q, etc., with the understanding that they are nondimensional.

Define

$$a = \frac{5}{4} \frac{1-2\nu}{1-\nu}$$

$$b = \frac{1}{4} \frac{1}{1-\nu}$$

$$c = \frac{21(1-\nu)}{1-2\nu}$$

$$d = \frac{7}{2} \frac{1}{1-2\nu} . \tag{C-16}$$

The differential equations can be written as

$$f'' + \frac{1}{r} f' - \frac{1}{r^2} f - af + bg' = 0 \tag{C-17a}$$

$$g'' + \frac{1}{r} g' - cg - d(f' + \frac{1}{r} f) = 0 . \tag{C-17b}$$

Consider

$$f = A\, I_1(kr)$$

$$g = B\, I_o(kr). \tag{C-18}$$

Where I's are modified Bessel functions of the first kind,

$$I_1' = k\, I_o(kr) - \frac{1}{r} I_i(kr)$$

$$I_1'' = k^2 I_1(kr) + \frac{2}{r^2} I_1(kr) - \frac{k}{r} I_o(kr)$$

$$I_o' = k\, I_1(kr)$$

$$I_o'' = k^2 I_o(kr) - \frac{k}{r} I_1(kr). \tag{C-19}$$

Substituting into equations (C-17a) and (C-17b),

$$[(k^2 - a)A + kbB]\, I_1 = 0 \tag{C-20a}$$

$$[(k^2 - c)B - kdA]\, I_o = 0 \tag{C-20b}$$

or

$$k^4 - (a + c - bd)k^2 + ac = 0. \tag{C-21}$$

It is noted that this equation is the same as equation (C-24); hence, the same k_1, k_2 will satisfy this equation and

$$F = -\frac{dk_i}{k_i^2 - c} E\ . \tag{C-22}$$

Now the general solution of equation (C-17) can be written as

$$f = A\, I_1(k_1 r) + B\, I_1(k_2 r) + E\, K_1(k_1 r) + F\, K_1(k, r) \tag{C-23}$$

$$g = \frac{dk_1}{k_1^2 - c} A\, I_o(k_1 r) + \frac{dk_2}{k_2^2 - c} B\, I_o(k_2 r) - E \frac{dk_1}{k_1^2 - c} K_o(k_1 r)$$

$$- \frac{dk_2}{k_2^2 - c} F\, K_o(k_2 r). \tag{C-24}$$

The derivatives are

$$f' = A[k_1 I_o(k_1 r) - \frac{1}{r} I_1(k_1 r)] + B[k_2 I_o(k_2 r) - \frac{1}{r} I_1(k_2 r)] - E[k_1 K_o(k_1 r) + \frac{1}{r} K_1(k_1 r)] - F[k_2 K_o(k_2 r) + \frac{1}{r} K_1(k_2 r)] \quad \text{(C-25)}$$

$$g' = \frac{dk_1}{k_1^2 - c} A\, I_1(k_1 r) + \frac{dk_2^2}{k_2^2 - c} B\, I_1(k_2 r) + \frac{dk_1^2}{k_1^2 - c} E\, K_1(k_1 r) + \frac{dk_2^2}{k_2^2 - c} F\, K_1(k_2 r). \quad \text{(C-26)}$$

Thus,

$$(k^2 - a)A + kbB = 0$$

$$-kdA + (k^2 - c)B = 0\,. \quad \text{(C-27)}$$

In order to have a solution, it is required that

$$\begin{pmatrix} k^2 - a & bk \\ -dk & k^2 - c \end{pmatrix} = 0 \quad \text{(C-28)}$$

or

$$k^4 - (a + c - bd)k^2 + ac = 0 \quad \text{(C-29)}$$

$$k^2 = \tfrac{1}{2}[(a + c - bd) \pm \sqrt{(a + c - bd)^2 - 4ac}]$$

$$k = \pm \{\tfrac{1}{2}(a + c - bd) \pm \tfrac{1}{2}\sqrt{(a + c - bd)^2 - 4ac}\}^{\frac{1}{2}}\,. \quad \text{(C-30)}$$

Since $I_p(kr) = -I_p(-kr)$, it is noted that there are only two independent solutions corresponding to

$$k_1 = \{\tfrac{1}{2}(a + c - bd) + \tfrac{1}{2}\sqrt{(a + c - bd)^2 - 4ac}\}^{\frac{1}{2}} \tag{C-31a}$$

and

$$k_2 = \{\tfrac{1}{2}(a + c - bd) - \tfrac{1}{2}\sqrt{(a + c - bd)^2 - 4ac}\}^{\frac{1}{2}} . \tag{C-31b}$$

In both cases,

$$B = \frac{k_i d}{k_i^2 - c} A. \tag{C-32}$$

Consider next

$$\begin{aligned} f &= E\, K_1(kr) \\ g &= F\, K_o(kr). \end{aligned} \tag{C-33}$$

Where K's are modified Bessel functions of the second kind,

$$\begin{aligned} K_1' &= -k\, K_o(kr) - \frac{1}{r} K_1(kr) \\ K_1'' &= k^2 K_1(kr) + \frac{2}{r^2} K_1(kr) + \frac{k}{r} K_o(kr) \\ K_o' &= -k\, K_1(kr) \\ K_o'' &= k^2 K_o(kr) + \frac{k}{r} K_1(kr). \end{aligned} \tag{C-34}$$

Substituting into equations (C-17),

$$\begin{aligned} [(k^2 - a)E - bkF]\, K_1 &= 0 \\ [(k^2 - c)F + dkE]\, K_o &= 0 . \end{aligned} \tag{C-35}$$

Thus,

$$(k^2 - a)E - bkF = 0$$

$$dkE + (k^2 - c)F = 0. \tag{C-36}$$

In order for this set to have a solution,

$$\begin{pmatrix} (k^2-a) & -bk \\ dk & (k^2-c) \end{pmatrix} = 0. \tag{C-37}$$

Substituting into equations (C-11),

$$\left(\frac{2}{7}\frac{dk_1^{\,2}}{k_1^2-c} - 1\right)[A\, I_1(k_1r_i) + E\, K_1(k_1r_i)] + \left(\frac{2}{7}\frac{dk_2^{\,2}}{k_2^2-c} - 1\right)[B\, I_1(k_2r_i)$$
$$+ F\, K_1(k_2r_i)] = 0$$

$$\left(\frac{2}{7}\frac{dk_1^{\,2}}{k_1^2-c} - 1\right)[A\, I_1(k_1r_o) + E\, K_1(k_1r_o)] + \left(\frac{2}{7}\frac{dk_2^{\,2}}{k_2^2-c} - 1\right)[B\, I_1(k_2r_o)$$
$$+ FK_1(k_2r_o)]=0$$

and

$$A[k_1I_o(k_1r_i) - \frac{1}{r_i} I_1(k_1r_i) + \frac{\nu}{1-\nu}\left(\frac{1}{r_i} I_1(k_1r_i) + \frac{1}{2}\frac{dk_1}{k_1^2-c} I_o(k_1r_i)\right)]$$

$$+ B[k_2I_o(k_2r_i) - \frac{1}{r_i} I_1(k_2r_i) + \frac{\nu}{1-\nu}\left(\frac{1}{r_i} I_1(k_2r_i) + \frac{1}{2}\frac{dk_2}{k_2^2-c} I_o(k_2r_i)\right)]$$

$$- E[k_1K_o(k_1r_i) + \frac{1}{r_i} K_1(k_1r_i) - \frac{\nu}{1-\nu}\left(\frac{1}{r_i} K_1(k_1r_i) - \frac{1}{2}\frac{dk_1}{k_1^2-c} K_o(k_1r_i)\right)]$$

$$- F[k_2K_o(k_2r_i) + \frac{1}{r_i} K_1(k_2r_i) - \frac{\nu}{1-\nu}\left(\frac{1}{r_i} K_1(k_2r_i) - \frac{1}{2}\frac{dk_2}{k_2^2-c} K_o(k_2r_i)\right)]$$
$$= \frac{5}{4}\frac{\nu}{1-\nu}\frac{w_o}{h} \tag{C-38}$$

$$A[k_1 I_o(k_1r_o) - \frac{1}{r_o} I_1(k_1r_o) + \frac{\nu}{1-\nu} (\frac{1}{r_o} I_1(k_1r_o) + \frac{1}{2} \frac{dk_1}{k_1^2-c} I_o(k_1r_o)]$$

$$+ B[k_2I_o(k_2r_o) - \frac{1}{r_o} I_1(k_2r_o) + \frac{\nu}{1-\nu} (\frac{1}{r_o} I_1(k_2r_o) + \frac{1}{2} \frac{dk_2}{k_2^2-c} I_o(k_2r_o)]$$

$$- E[k_1K_o(k_1r_o) + \frac{1}{r_o} K_1(k_1r_o) - \frac{\nu}{1-\nu} (\frac{1}{r_o} K_1(k_1r_o) - \frac{1}{2} \frac{dk_1}{k_1^2-c} K_o(k_1r_o)]$$

$$- F[k_2K_o(k_2r_o) + \frac{1}{r_o} K_1(k_2r_o) - \frac{\nu}{1-\nu} (\frac{1}{r_o} K_1(k_2r_o) - \frac{1}{2} \frac{dk_2}{k_2^2-c} K_o(k_2r_o)]$$

$$= - \frac{5}{4} (\frac{\nu}{1-\nu}) \frac{w_o}{h}. \qquad \text{(C-39)}$$

For very large values of kr, the modified Bessel functions have the asymptotic behavior

$$I_\rho(kr) \sim \frac{e^{kr}}{\sqrt{2\pi kr}} \qquad (kr \to \infty) \qquad \text{(C-40)}$$

$$I_\rho(kr) \sim \frac{e^{-kr}}{\sqrt{\frac{2}{\pi}kr}} .$$

Using this and a set of new constants defined by

$$A' = e^{k_1r_o} \frac{1}{\sqrt{2\pi k_1r_o}} A \qquad B' = e^{k_2r_o} \frac{1}{\sqrt{2\pi k_2r_o}} B$$

$$E' = e^{-k_1r_i} \frac{1}{\sqrt{\frac{2}{\pi}k_1r_u}} E \qquad F' = e^{-k_2r_i} \frac{1}{\sqrt{\frac{2}{\pi} k_2r_i}} F, \qquad \text{(C-41)}$$

the four boundary conditions are written as

$$A'\left(\frac{2}{7}\frac{dk_1{}^2}{k_1^2-c}-1\right)\sqrt{\frac{r_o}{r_i}}\,e^{k_1(r_i-r_o)}+B'\left(\frac{2}{7}\frac{dk_2{}^2}{k_2^2-c}-1\right)\sqrt{\frac{r_o}{r_i}}\,e^{k_2(r_i-r_o)}$$

$$+E'\left(\frac{2}{7}\frac{dk_1{}^2}{k_1^2-c}-1\right)+F'\left(\frac{2}{7}\frac{dk_2{}^2}{k_2^2-c}-1\right)=0 \qquad \text{(C-42a)}$$

$$A'\left(\frac{2}{7}\frac{dk_1{}^2}{k_1^2-c}-1\right)+B'\left(\frac{2}{7}\frac{dk_2{}^2}{k_2^2-c}-1\right)+E'\left(\frac{2}{7}\frac{dk_1{}^2}{k_1^2-c}-1\right)\sqrt{\frac{r_i}{r_o}}\,e^{-k_1(r_o-r_i)}$$

$$+F'\left(\frac{2}{7}\frac{dk_2{}^2}{k_2^2-c}-1\right)\sqrt{\frac{r_i}{r_o}}\,e^{-k_2(r_o-r_i)}=0 \qquad \text{(C-42b)}$$

$$A'\left[k_1-\frac{1}{r_i}+\frac{\nu}{1-\nu}\left(\frac{1}{r_i}+\frac{1}{2}\frac{dk_1}{k_1^2-c}\right)\right]\sqrt{\frac{r_o}{r_i}}\,e^{k_1(r_i-r_o)}+B'\left[k_2-\frac{1}{r_i}+\frac{\nu}{1-\nu}\right.$$

$$\left.+\frac{1}{2}\frac{kd_2}{k_2^2-c}\right)\Big]\sqrt{\frac{r_o}{r_i}}\,e^{k_2(r_i-r_o)}-E'\left[k_1+\frac{1}{r_i}-\frac{\nu}{1-\nu}\left(\frac{1}{r_i}-\frac{dk_1}{k_1^2-c}\right)\right]$$

$$-F'\left[k_2+\frac{1}{r_i}-\frac{\nu}{1-\nu}\left(\frac{1}{r_i}-\frac{1}{2}\frac{dk_2}{k_2^2-c}\right)\right]=-\frac{5}{4}\left(\frac{\nu}{1-\nu}\right)\frac{w_o}{h} \qquad \text{(C-42c)}$$

$$A'[k_1 - \frac{1}{r_o} + \frac{\nu}{1-\nu}(\frac{1}{r_o} + \frac{1}{2}\frac{dk_1}{k_1^2-c})] + B'[k_2 - \frac{1}{r_o} + \frac{\nu}{1-\nu}(\frac{1}{r_o} + \frac{1}{2}\frac{dk_2}{k_2^2-c})]$$

$$- E'[k_1 + \frac{1}{r_o} - \frac{\nu}{1-\nu}(\frac{1}{r_o} - \frac{1}{2}\frac{dk_1}{k_1^2-c})]\sqrt{\frac{r_i}{r_o}}\; e^{-k_1(r_o-r_i)} - F'[k_2$$

$$+ \frac{1}{r_o} - \frac{\nu}{1-\nu}(\frac{1}{r_o} - \frac{1}{2}\frac{dk_2}{k_2^2-c})]\sqrt{\frac{r_i}{r_o}}\; e^{-k_2(r_o-r_i)} = -\frac{5}{4}(\frac{\nu}{1-\nu})\frac{w_o}{h}. \qquad \text{(C-42d)}$$

For large values of r_o-r_i, A', B' in equations (C-42a and c) and E', F' in equations (C-42b and d) can be disregarded.

Hence,

$$E' = -\frac{\frac{2}{7}\frac{dk_2^{\,2}}{k_2^2-c} - 1}{\frac{2}{7}\frac{dk_1^{\,2}}{k_1^2-c} - 1}\, F',$$

$$E'[k_1 + \frac{1}{r_i} - \frac{\nu}{1-\nu}(\frac{1}{r_i} - \frac{dk_1}{k_1^2-c})] + F'[k_2 + \frac{1}{r_i} - \frac{\nu}{1-\nu}(\frac{1}{r_i} - \frac{1}{2}\frac{dk_2}{dk_2^2-c})]$$

$$= \frac{5}{4}(\frac{\nu}{1-\nu})\frac{w_o}{h}, \qquad \text{(C-43)}$$

and

$$A' = -\frac{\frac{2}{7}\frac{dk_2^{\,2}}{k_2^2-c} - 1}{\frac{2}{7}\frac{dk_1^{\,2}}{k_1^2-c} - 1}\, B',$$

$$A'[k_1 - \frac{1}{r_o} + \frac{\nu}{1-\nu}(\frac{1}{r_o} + \frac{1}{2}\frac{dk_1}{k_1^2-c})] + B'[k_2 - \frac{1}{r_o} + \frac{\nu}{1-\nu}(\frac{1}{r_o} + \frac{1}{2}\frac{dk_2}{k_2^2-c})]$$

$$= -\frac{5}{4}(\frac{\nu}{1-\nu})\frac{w_o}{h}. \qquad \text{(C-44)}$$

In terms of A's, the general solution can be written as

$$f = A'\sqrt{\frac{r_o}{r}}\, e^{k_1(r-r_o)} + B'\sqrt{\frac{r_o}{r}}\, e^{k_2(r-r_o)} + E'\sqrt{\frac{r_o}{r}}\, e^{-k_1(r-r_o)}$$

$$+ F'\sqrt{\frac{r_o}{r}}\, e^{-k_2(r-r_o)}$$

$$g = \frac{dk_1}{k_1^2-c} A'\sqrt{\frac{r_o}{r}}\, e^{k_1(r-r_o)} + \frac{dk_2}{k_2^2-c} B'\sqrt{\frac{r_o}{r}}\, e^{k_2(r-r_o)}$$

$$- E'\frac{dk_1}{k_1^2-c}\sqrt{\frac{r_o}{r}}\, e^{-k_1(r-r_i)} - F'\frac{dk_2}{k_2^2-c}\sqrt{\frac{r_i}{r}}\, e^{-k_2(r-r_i)}. \qquad \text{(C-45)}$$

When $r_o - r_i$ is large, the solution can be further simplified:

(1) Region near r_o,

$$f = A'\sqrt{\frac{r_o}{r}}\, e^{k_1(r-r_o)} + B'\sqrt{\frac{r_o}{r}}\, e^{k_2(r-r_o)}$$

$$g = A'\frac{dk_1}{k_1^2-c}\sqrt{\frac{r_o}{r}}\, e^{k_1(r-r_o)} + B'\frac{dk_2}{k_2^2-c}\sqrt{\frac{r_o}{r}}\, e^{k_2(r-r_o)}. \qquad \text{(C-46)}$$

(2) Region near r_i,

$$f = E'\sqrt{\frac{r_i}{r}}\, e^{-k_1(r-r_i)} + F'\sqrt{\frac{r_i}{r}}\, e^{-k_2(r-r_i)}$$

$$g = -E'\frac{dk_1}{k_1^2-c}\sqrt{\frac{r_i}{r}}\, e^{-k_1(r-r_i)} - F'\frac{dk_2}{k_2^2-c}\sqrt{\frac{r_i}{r}}\, e^{-k_2(r-r_i)}. \qquad \text{(C-47)}$$

(3) Away from ends,

$$f = 0$$

$$g = 0\,. \qquad \text{(C-48)}$$

Calculation of stresses is

$$\sigma_z = (\lambda + 2G)\frac{\partial w}{\partial z} + \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)$$

$$\sigma_{rz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \tag{C-50}$$

at points on bonded surfaces ($z = \pm h$)

$$\frac{\partial u}{\partial r} = \frac{u}{r} = 0$$

$$\frac{\partial w}{\partial r} = 0 \;;$$

thus, at $z = h$,

$$\sigma_z = (\lambda + 2G)\frac{\partial w}{\partial z}$$

$$= \frac{2G(1-\nu)}{1-2\nu}\left(\frac{w_o}{h} - 2g\right)$$

$$\sigma_{rz} = -\,2G\,f. \tag{C-51}$$

Average σ_z at bonded surface is

$$\sigma_{z_{avg}}\Big|_{z=+h} = \frac{2}{r_o^2 - r^2}\int_{r_i}^{r_o} \sigma_z\Big|_{z=+h} r\,dr$$

$$= 2\left(\frac{2G(1-\nu)}{1-2\nu}\right)\left(\frac{1}{r_o^2 - r_i^2}\right)\int_{r_i}^{r_o}\frac{\partial w}{\partial z}\, r\,dr$$

$$= \left(\frac{2G(1-\nu)}{1-2\nu}\right)\left\{\left(\frac{wo}{h}\right) - \frac{4}{r_o^2 - r_i^2}\int_{r_i}^{r_o} g\,r\,dr\right\}, \tag{C-52}$$

since

$$g = \frac{dk_1}{k_1^2-c} [A\ I_o(k_1 r) - E\ K_o(k_1 r)]$$

$$+ \frac{dk_2}{k_2^2-c} [B\ I_o(k_2 r) - F\ K_o(k_2 r)]$$

$$\int_{r_i}^{r_o} grdr = \{\frac{d}{k_1{}^2-c} [Ar\ I_1(k_1 r) + Er\ K_1(k_1 r)]$$

$$+ \frac{dr}{k_2^2-c} [B\ I_1(k_2 r) + F\ K_1(k_2 r)]\}\Big|_{r_o}^{r_o}$$

$$= r_o \{\frac{d}{k_1{}^2-c} [A\ I_1(k_1 r_o) + E\ K_1(k_1 r_o)] + \frac{d}{k_2{}^2-c} [BI_1(k_2 r_o)$$

$$+ F\ K_1(k_2 r_c)]\}$$

$$- r_i \{\frac{d}{k_1{}^2-c} [A\ I_1(k_1 r_i) + E\ K_1(k_1 r_i)] + \frac{d}{k_2{}^2-c} [B\ I_1(k_2 r_i)$$

$$+ F\ K_1(k_1 r_i)]\}.$$

For very large values of kr,

$$\int_{r_i}^{r_o} grdr = r_o \{\frac{d}{k_1{}^2-c} [A \frac{e^{+k_1 r_o}}{\sqrt{2\pi k_1 r_o}} + E \frac{e^{-k_1 r_o}}{\sqrt{\frac{2}{\pi} k_1 r_o}}] + \frac{d}{k_2{}^2-c} [B \frac{e^{k_1 r_o}}{\sqrt{2\pi k_2 r_o}}$$

$$+ F \frac{e^{-k_2 r_o}}{\sqrt{\frac{2}{\pi} k_2 r_o}}]\}$$

$$- r_i \{\frac{d}{k_1^{\,2}-c} [A \frac{e^{k_1 r_i}}{\sqrt{2\pi k_1 r_i}} + E \frac{e^{-k_1 r_i}}{\sqrt{\frac{2}{\pi} k_1 r_i}}] + \frac{d}{k_2^{\,2}-c} [B \frac{e^{k_2 r_i}}{\sqrt{2\pi k_2 r_i}}$$

$$+ F \frac{e^{-k_2 r_i}}{\sqrt{\frac{2}{\pi} k_2 r_i}}$$

$$= r_o \{\frac{d}{k_1^{\,2}-c} (A' + E' \sqrt{\frac{r_i}{r_o}}\, e^{-k_1 (r_o - r_i)} + \frac{d}{k_2^{\,2}-c} [B'$$

$$+ F' \sqrt{\frac{r_i}{r_o}} e^{-k(r_o - r_i)}]\}$$

$$- r_i \{\frac{d}{k_1^{\,2}-c} (A' \sqrt{\frac{r_o}{r_i}}\, e^{-k_1 (r_o - r_i)} + E')$$

$$+ \frac{d}{k_2^2-c} [B' \sqrt{\frac{r_o}{r_i}}\, e^{-k_1 (r_o - r_i)} + F']\}$$

or

$$\int_{r_i}^{r_o} grdr = r_o \left(\frac{d}{k_1^{\,2}-c} A' + \frac{d}{k_1^{\,2}-c} B'\right) - r_i \left(\frac{d}{k_1^{\,2}-c} E' + \frac{d}{k_1^{\,2}-c} F'\right)$$

and

$$\sigma_{z_{max}} \Big|_{z=h} = \frac{2G(1-\nu)}{1-2\nu} \left(\frac{w_o}{h}\right) ; \qquad \text{(C-53)}$$

hence,

$$\frac{\sigma_{z\ max}}{\sigma_{z\ avg}} = \frac{\left(\frac{w_o}{h}\right)}{\frac{w_o}{h} \left(\frac{4}{r_o^{\,2} - r_i^{\,2}}\right) \int_{r_i}^{r_o} grdr} . \qquad \text{(C-54)}$$

It is also noted that the maximum shear stress occurs at the outer edge; i. e., $r = r_o$.

Hence,

$$\frac{\sigma_{rz\ max}}{\sigma_{z\ avg}} = \frac{+\ 2G\ f(r_o)}{\frac{2G(1-\nu)}{1-2\nu}\left(\frac{w_o}{h} - \frac{4}{r_o^2-r_i^2}\int_{r_i}^{r_o} grdr\right)}$$

$$= \frac{f(r_o)}{\frac{1-\nu}{1-2\nu}\left(\frac{w_o}{h} - \frac{4}{r_o^2-r_i^2}\int_{r_i}^{r_o} grdr\right)} \quad \text{(C-55)}$$

This is programmed for the computer as follows:

Let

$$\bar{f} = X1 \qquad \bar{f}' = Z1$$

$$\bar{g} = X2 \qquad \bar{g}' = Z2.$$

The differential equations can be written as

$$(Z1)' + \frac{1}{r} Z1 - \frac{1}{r^2} X1 - \frac{5}{4}\frac{1-2\nu}{1-\nu} X1 + \frac{1}{4}\frac{1}{1-\nu} Z2 = 0$$

$$(X1)' = Z1$$

$$(Z2)' + \frac{1}{r} Z2 - \frac{21(1-\nu)}{1-2\nu} X2 - \frac{7}{2(1-2\nu)}\left(Z1 + \frac{1}{r} X1\right) = 0$$

$$(X2)' = Z2 .$$

Define

$$AA = \frac{1-2\nu}{1-\nu}$$

$$BB = \frac{1}{1-\nu}$$

$$CC = \frac{1}{1-2\nu}$$

DRAD = (RADN - RADØ)/FNR

RAD = RADØ + FI*RAD   FI = I - 1

NRR = (NR/NP) + 1.0

ND - number of intervals skipped during printing

$$(Z1)' = \frac{1}{RAD} Z1 + \frac{1}{RAD^2} X1 + (1.25*AA)*(X1) - (0.25*BB)*z2$$

$$(X1)' = Z1$$

$$(Z1)' = 1\frac{1}{RAD} Z2 + \frac{21}{AA} *Z2 + 3.5*CC*(Z1 + \frac{X1}{RAD})$$

$$(X2)' = Z2 .$$

Define

$$DERZ1F\ (A,B,C,D) = DRAD*\left(-\frac{A}{D} + \frac{B}{D*D} + (1.25*AA)*B - 0.25*BB*C\right)$$

$$DERX1F\ (A) = DRAD*A$$

$$DERZ2F\ (A,B,C,D,E) = DRAD*\left(-\frac{C}{E} + \frac{21}{AA} *D+3.5*CC*(A + \frac{B}{E})\right)$$

$$DERX2F\ (A) = DRAD*A .$$

The four independent solutions of X1, Z1, X2, Z2 are

X1	S2(1,IJ),	S2(2,IJ),	S2(3,IJ),	S2(4,IJ)
Z1	S1(1,IJ),	S1(2,IJ),	S1(3,IJ),	S1(4,IJ)
X2	S4(1,IJ),	S4(2,IJ),	S4(3,IJ),	S4(4,IJ)
Z2	S3(1,IJ),	S3(2,IJ),	S3(3,IJ),	S3(4,IJ).

Where the first index denotes number of independent solutions,

F(IJ) = A(1) S2(1,IJ) + A(2) S2(2,IJ) + A(3) S2(3,IJ) + A(4) S2(4,I) = X1

FF(IJ) = A(1) S1(1,IJ) + A(2) S1(2,IJ) + A(3) S1(3,IJ) + A(4) S1(4,IJ) = Z1

G(IJ) = A(1) S4(1,IJ) + A(2) S4(2,IJ) + A(3) S4(3,IJ) + A(4) S4(4,IJ) = X2

GG(IJ) = A(1) S3(1,IJ) + A(2) S3(2,IJ) + A(3) S3(3,IJ) + A(4) S3(4,IJ) = Z2.

Since the boundary condition states

$$\bar{f}' + \left(\frac{\nu}{1-\nu}\right)\left(\frac{\bar{f}}{\bar{r}_i} + \tfrac{1}{2}\,\bar{g}\right) = -\frac{5}{4}\left(\frac{\nu}{1-\nu}\right)\frac{w_o}{h}$$

$$\bar{f}' + \left(\frac{\nu}{1-\nu}\right)\left(\frac{\bar{f}}{\bar{r}_o} + \tfrac{1}{2}\,\bar{g}\right) = -\frac{5}{4}\left(\frac{\nu}{1+\nu}\right)\frac{w_o}{h}$$

$$2/7\ \bar{g}' - \bar{f} = 0$$

$$2/7\ \bar{g}' - \bar{f} = 0,$$

using

DD = GNU/(1.0 - GNU)

$W\emptyset = w_o/h$

and

F, FF, G, GG,

the following are obtained:

$$\left(S1(1,1) + DD*\left(\frac{S2(1,1)}{RAD\emptyset} + 0.5*S4(1,1)\right)\right) A_1$$

$$+\left(S1(2,1) + DD*\left(\frac{S2(2,1)}{RAD\emptyset} + 0.5*S4(2,1)\right)\right) A_2$$

$$+\left(S1(3,1) + DD*\left(\frac{S2(3,1)}{RAD\emptyset} + 0.5*S4(3,1)\right)\right) A_3$$

$$+\left(S1(4,1) + DD*\left(\frac{S2(4,1)}{RAD\emptyset} + 0.5*S4(4,1)\right)\right) A_4 = -\ DD*W\emptyset*1.25$$

$$\left(S1(1,NRR) + DD*\left(\frac{S2(1,NRR)}{RADN} + 0.5*\ S4(1,NRR)\right)\right) A_1$$

$$+\left(S1(2,NRR) + DD*\left(\frac{S2(2,NRR)}{RADN} + 0.5*\ S4(2,NRR)\right)\right) A_2$$

$$+\left(S1(3,NRR) + DD*\left(\frac{S2(3,NRR)}{RADN} + 0.5*\ S4(3,NRR)\right)\right) A_3$$

$$+\left(S1(4,NRR) + DD*\left(\frac{S2(4,NRR)}{RADN} + 0.5*\ S4(4,NRR)\right)\right) A_4 = -\ DD*\ 1.25*\ W\emptyset$$

$$\left(\frac{2}{7}\,S3(1,1) - S2(1,1)\right)A_1 + \left(\frac{2}{7}\,S3(2,1) - S2(Z,1)\right)A_2$$

$$+ \left(\frac{2}{7}\,S3(3,1) - S2(3,1)\right)A_3 + \left(\frac{2}{7}\,S3(4,1) - S2(4,1)\right)A_4 = 0$$

$$\left(2/7\ S3(1,NRR) - S2(1,NRR)\right)A_1 + \left(2/7\ S3(2,NRR) - S2(2,NRR)\right)A_2$$

$$+ \left(2/7\ S3(3,NRR) - S2(3,NRR)\right)A_3 + \left(2/7\ S3(4,NRR) - S2(4,NRR)\right)A_4 = 0.$$

Let these equations be represented as

$$\{TRX\}\ \{A\} = \{B\}\ .$$

Solving for A,

$$\{TRX\}^{-1}\ \{B\} = \{A\}.$$

Since B can be written as

$$-\ DD{*}\ 1.25{*}\ w\emptyset \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix},$$

$$\{A\} = -\ DD{*}\ 1.25{*}\ w\emptyset\ \{TRX\}^{-1} \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}.$$

Calculation of stresses is

$$\sigma_{ij} = \lambda\Delta\delta_{ij} + 2G\ \epsilon_{ij}$$

$$\sigma_z = \lambda\Delta + 2G\ \epsilon_z$$

$$= (\lambda+2G)\epsilon_z + \lambda(\epsilon_\theta + \epsilon_r)$$

$$= (\lambda+2G)\ \frac{\partial w}{\partial z} + \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right).$$

$$\sigma_{rz} = 2G\epsilon_{rz}$$

$$= G\ \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right).$$

Consider points on flat surface ($z = \pm h$),

$$\frac{\partial u}{\partial r} = \frac{u}{r} = 0$$

$$\frac{\partial w}{\partial r} = 0 \cdot$$

Thus,

$$\sigma_z = (\lambda+2G)\,\frac{\partial w}{\partial z} = (\lambda+2G)\,[\frac{w_o}{h} - 2g(r)]$$

$$= (\lambda+2G)\,(w\phi - 2G) = \frac{2G(1-\nu)}{1-2\nu}\,(w\dot{\phi} - 2G)$$

$$\sigma_{rz} = G\,\frac{\partial u}{\partial z} = -\,2G\,\frac{f(r)}{h} = -\,2G\,\bar{f}\ .$$

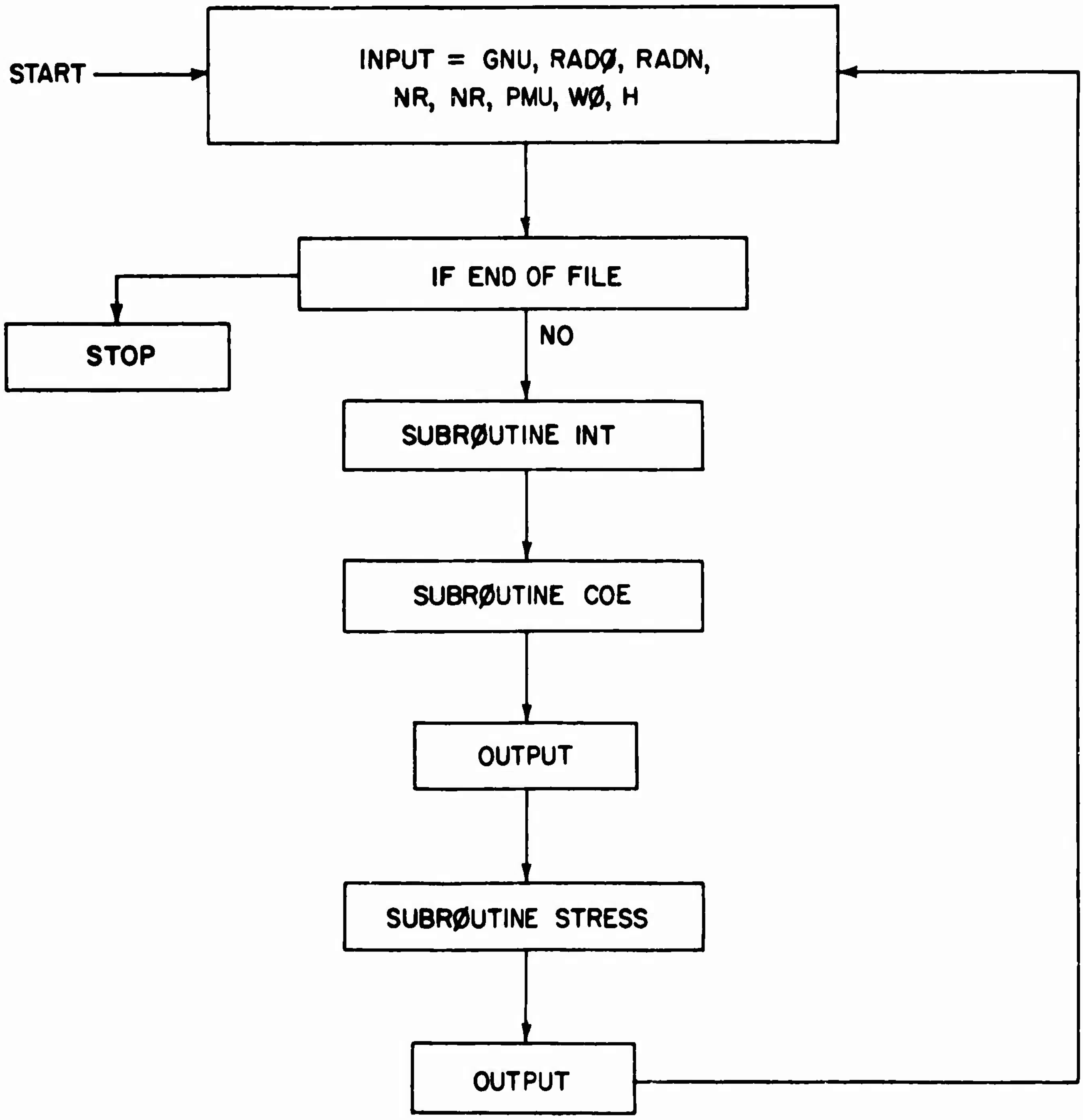


Figure 11. Main Flow Chart.

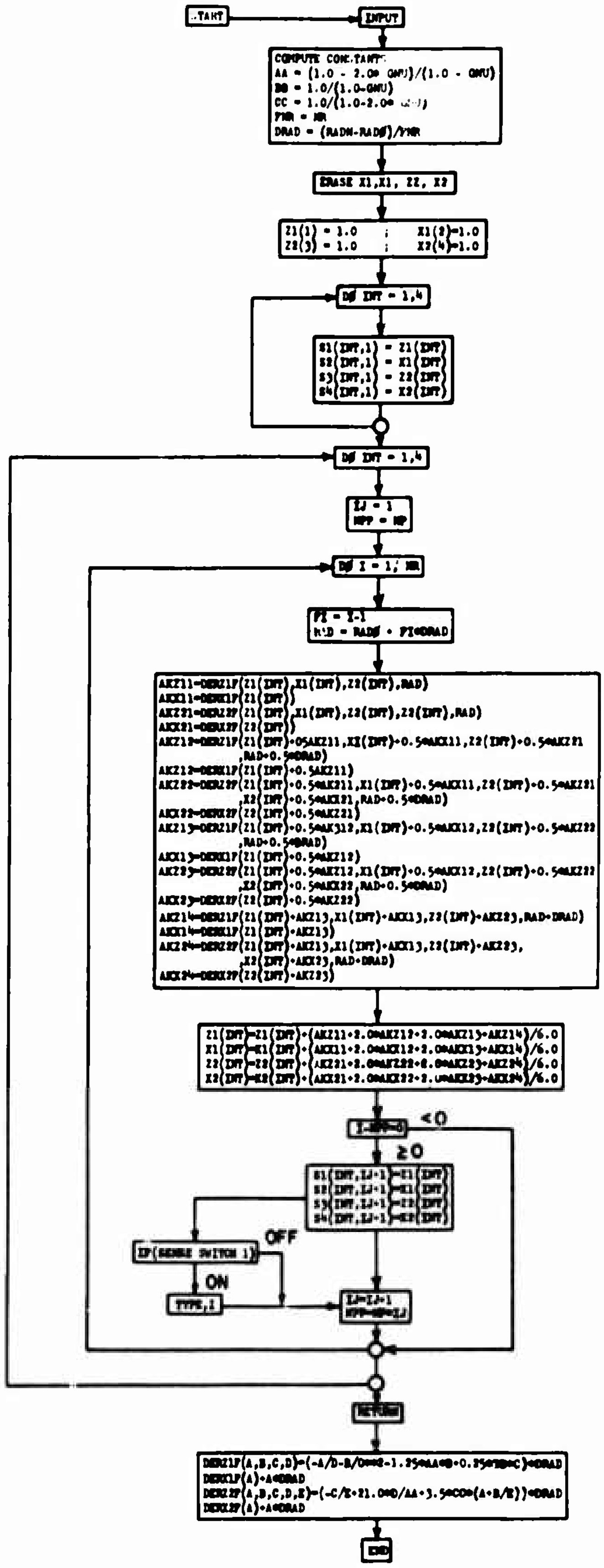


Figure 12. Subroutine Int.

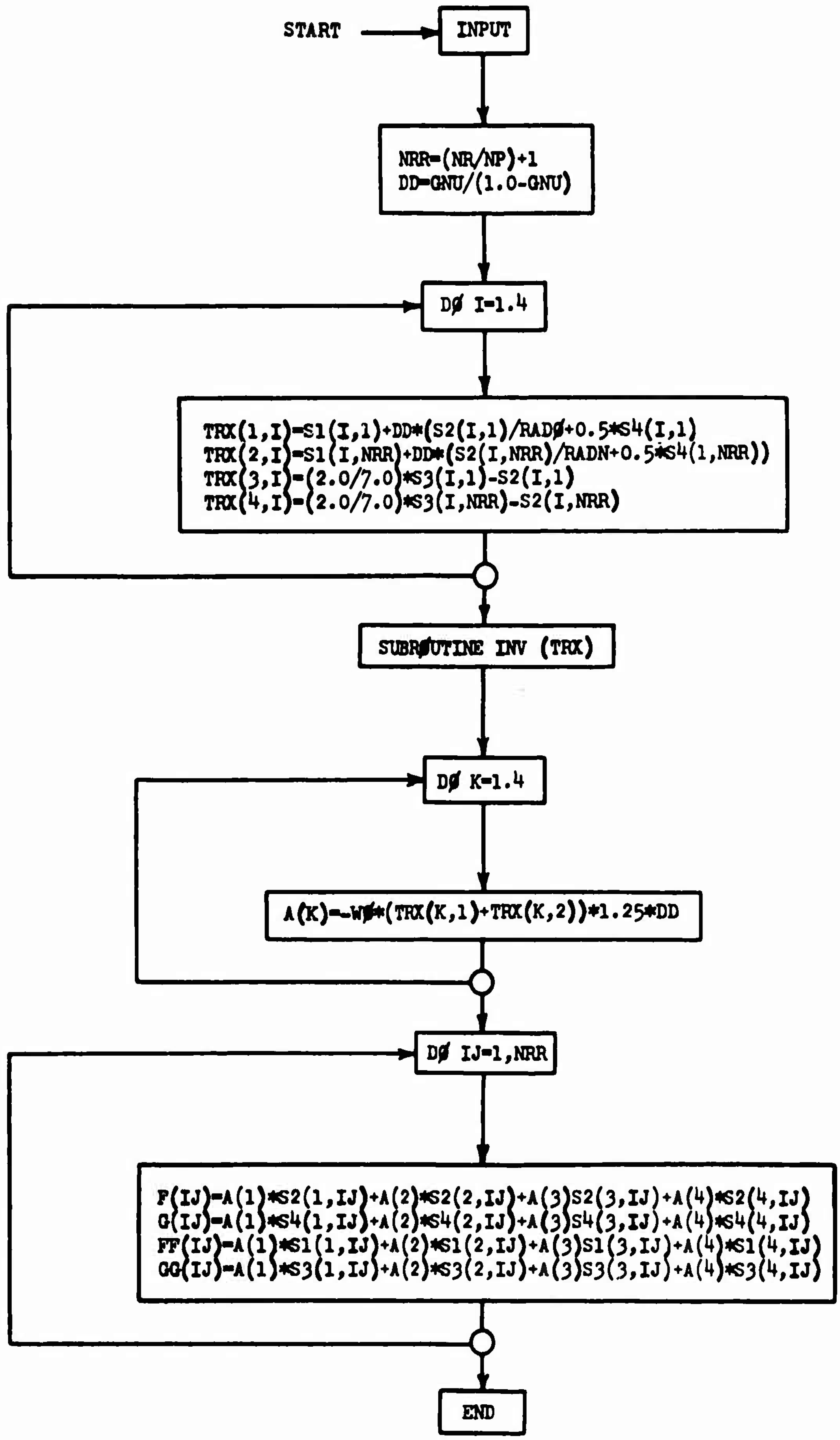


Figure 13. Subroutine Coe.

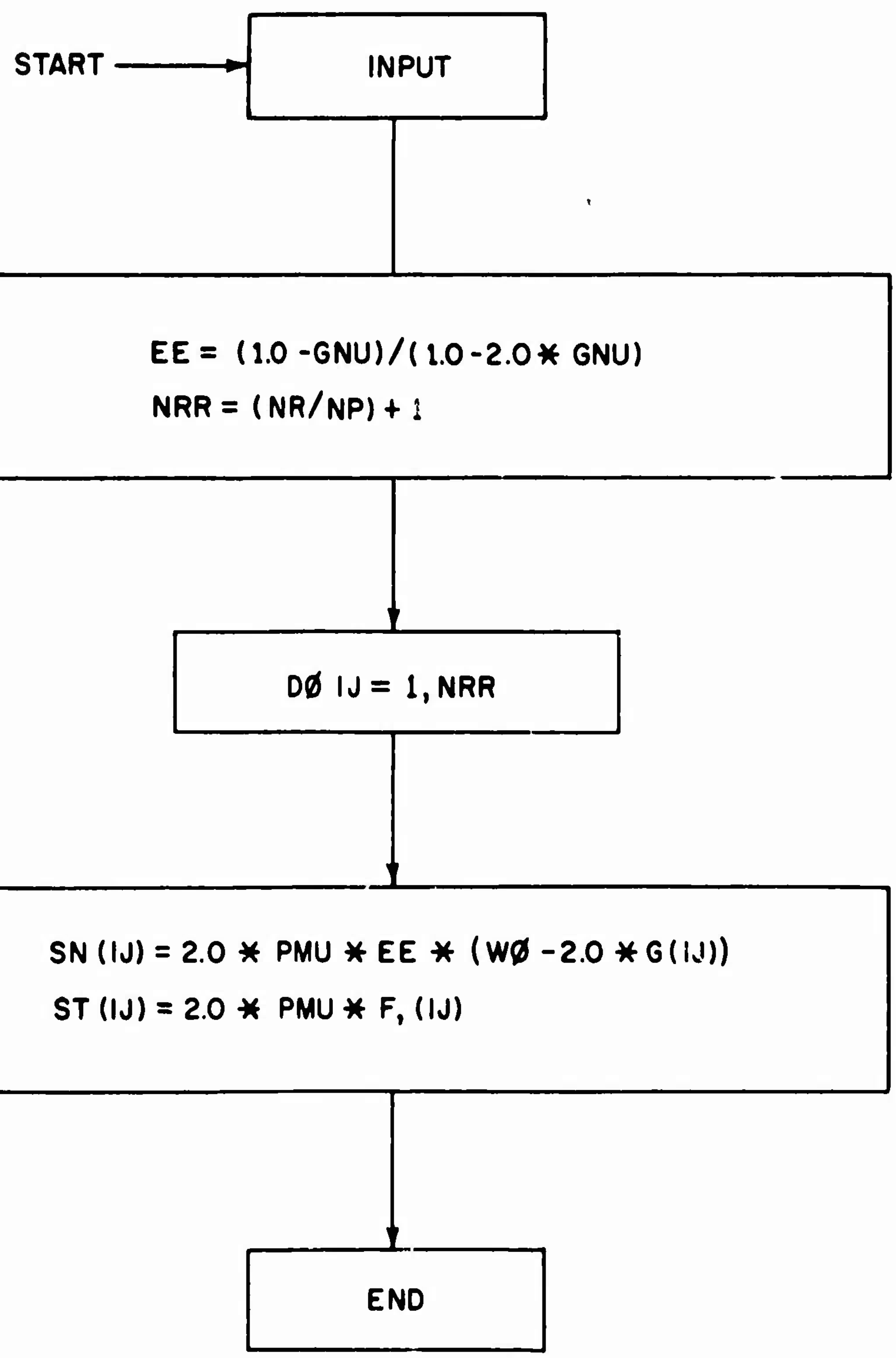


Figure 14. Subroutine Stress.

*** S O U R C E   P R O G R A M   L I S T I N G ***

```
TITLEBEARING
C     MAIN PROGRAM  LAMINATED BEARING
      COMMON  GNU, RADO, RADN, NR, NP, PMU, WO, H,X1,Z1,X2,Z2,
     1S1, S2, S3, S4, F, FF, G, GG, SN, ST
      DIMENSION  X1(4),Z1(4),Z2(4),X2(4), S1(4,51 ),S2(4,51 ),S3(4,51),
     1S4(4,51 ), F(51), FF(51), G(51), GG(51), SN(51), ST(51)
   10 READ 4,GNU,RADO,RADN,NR,NP,PMU,WO,H
    4 FORMAT (3E15.8, 2I5/3E15.8)
      IF END OF FILE 200,201
  200 STOP
  201 PRINT 6, GNU,RADO,RADN,NR
    6 FORMAT (1H1/3X,6HGNU = F10.5, 3X,7HRADO = F10.3, 3X,7HRADN = F10.3
     1, 3X,5HNR = I5//)
      PRINT 7,NP, PMU, WO, H
    7 FORMAT (3X,5HNP = I5, 3X,6HPMU = F10.5, 3X,5HWO = F10.5,
     13X,4HH = F10.8////)
   20 CALL INT
   40 CALL COE
      PRINT 1
    1 FORMAT (1H1/45X,29HDISPLACEMENTS AND DERIVATIVES//)
      PRINT 2
    2 FORMAT (// 18X,2HIJ 6X,11HF  FUNCTION 7X,11HG  FUNCTION 18X,
     113H F DERIVATIVE  7X,13H G DERIVATIVE//)
      NRR =(NR/NP)+1
      PRINT 3,(IJ, F(IJ), G(IJ), FF(IJ), GG(IJ),IJ=1,NRR)
    3 FORMAT (18X,I2,2X,E15.8,3X,E15.8, 16X,E15.8,5X,E15.8)
   60 CALL STRESS
      PRINT 101
  101 FORMAT (1H1/47X 23H STRESS AT FLAT SURFACE)
      PRINT 102
  102 FORMAT (// 35X 2HIJ 7X 14H NORMAL STRESS 10X 14H SHEAR  STRESS)
      PRINT 103, (IJ, SN(IJ), ST(IJ), IJ=1,NRR)
  103 FORMAT (35X I2, 6X E15.8, 9X E15.8)
      GO TO 10
      END
```

```
      SUBROUTINEINT
      COMMON  GNU, RADO, RADN, NR, NP, PMU, WO, H,X1,Z1,X2,Z2,
     1S1, S2, S3, S4, F, FF, G, GG, SN, ST
      DIMENSION  X1(4),Z1(4),Z2(4),X2(4), S1(4,51 ),S2(4,51 ),S3(4,51),
     1S4(4,51 ), F(51),  FF(51), G(51), GG(51), SN(51), ST(51)
      AA = (1.0 - 2.0*GNU)/(1.0 - GNU)
      BB = 1.0 /(1.0 - GNU)
      CC = 1.0 /(1.0 - 2.0*GNU)
      FNR = NR
      DRAD = (RADN - RADO)/FNR
      ERASE (Z1,X1,Z2,X2)
      Z1(1)   =1.0
      X1(2)   = 1.0
      Z2(3)   = 1.0
      X2(4)   = 1.0
      DO 20 INT = 1,4
      S1(INT,1) = Z1(INT)
      S2(INT,1) = X1(INT)
      S3(INT,1) = Z2(INT)
      S4(INT,1) = X2(INT)
   20 CONTINUE
      DO 100  INT=1,4
      IJ=1
      NPP=NP
      DO 60 I= 1,NR
      FI = I -1
      RAD = RADO + FI*DRAD
      AKZ11 = DERZ1F(Z1(INT),X1(INT),Z2(INT),RAD)
      AKX11 = DERX1F(Z1(INT))
      AKZ21 = DERZ2F(Z1(INT)   , X1(INT)   ,Z2(INT)   ,X2(INT)   , RAD)
      AKX21 = DERX2F(Z2(INT))
      AKZ12 = DERZ1F(Z1(INT) + 0.5*AKZ11,X1(INT) + 0.5*AKX11, Z2(INT)
     1+0.5*AKZ21, RAD+0.5*DRAD)
      AKX12 = DERX1F(Z1(INT) + 0.5*AKZ11)
      AKZ22 = DERZ2F(Z1(INT) + 0.5*AKZ11, X1(INT) + 0.5*AKX11,Z2(INT)
```

A

```
      AKX21 = DERX2F(Z2(INT))
      AKZ12 = DERZ1F(Z1(INT) + 0.5*AKZ11,X1(INT) + 0.5*AKX11, Z2(INT)
     1+0.5*AKZ21, RAD+0.5*DRAD)
      AKX12 = DERX1F(Z1(INT) + 0.5*AKZ11)
      AKZ22 = DERZ2F(Z1(INT) + 0.5*AKZ11, X1(INT) + 0.5*AKX11,Z2(INT)
     1+0.5*AKZ21, X2(INT) + 0.5*AKX21, RAD+0.5*DRAD)
      AKX22 = DERX2F(Z2(INT) + 0.5*AKZ21)
      AKZ13 = DERZ1F(Z1(INT) + 0.5*AKZ12, X1(INT) + 0.5*AKX12, Z2(INT)
     1+0.5*AKZ22, RAD+0.5*DRAD)
      AKX13 = DERX1F(Z1(INT) + 0.5*AKZ12)
      AKZ23 = DERZ2F(Z1(INT) + 0.5*AKZ12, X1(INT) + 0.5*AKX12, Z2(INT)
     1+0.5*AKZ22, X2(INT) + 0.5*AKX22,RAD+0.5*DRAD)
      AKX23 = DERX2F(Z2(INT) + 0.5*AKZ22)
      AKZ14 = DERZ1F(Z1(INT) + AKZ13, X1(INT) + AKX13, Z2(INT) + AKZ23,
     1RAD+DRAD)
      AKX14 = DERX1F(Z1(INT) + AKZ13)
      AKZ24 = DERZ2F(Z1(INT) + AKZ13, X1(INT) + AKX13, Z2(INT) + AKZ23,
     1X2(INT) + AKX23, RAD+DRAD)
      AKX24 = DERX2F(Z2(INT) + AKZ23)
      Z1(INT)     = Z1(INT)   + (AKZ11+2.0*AKZ12+2.0*AKZ13+AKZ14)/6.0
      X1(INT)     = X1(INT)   + (AKX11+2.0*AKX12+2.0*AKX13+AKX14)/6.0
      Z2(INT)     = Z2(INT)   + (AKZ21+2.0*AKZ22+2.0*AKZ23+AKZ24)/6.0
      X2(INT)     = X2(INT)   + (AKX21+2.0*AKX22+2.0*AKX23+AKX24)/6.0
      IF(I-NPP)60,30,30
   30 S1(INT,IJ+1) = Z1(INT)
      S2(INT,IJ+1) = X1(INT)
      S3(INT,IJ+1) = Z2(INT)
      S4(INT,IJ+1) = X2(INT)
      IF (SENSE SWITCH 1) 200,300
  200 TYPE,I
  300 CONTINUE
      IJ=IJ+1
      NPP= NP*IJ
   60 CONTINUE
  100 CONTINUE
      RETURN
      DERZ1F(A,B,C,D) = (-A/D+ B/D**2 + 1.25*AA*B - 0.25*BB*C)*DRAD
      DERX1F(A) = A * DRAD
      DERZ2F(A,B,C,D,E) = (-C/E + 21.0*D/AA + 3.5*CC*(A + B/E))*DRAD
      DERX2F(A) = A * DRAD
      END
```

B

*** SOURCE PROGRAM LISTING ***

```
      SUBROUTINE  INV(A)
C ROUTINE TO INER  A 4 BY 4 MATRIX, A(4, 4)
      DIMENSION A(4, 4)
      DO 5 I = 1, 4
      B = A(I, I)
      DO 1 K = 1, 4
    1 A(I, K) = A(I, K)/B
      DO 4 K = 1, 4
      IF(K-I) 2, 4, 2
    2 C = A(K, I)
      DO 3 L = 1, 4
    3 A(K, L) = A(K, L) - C * A(I, L)
      A(K, I) = - C / B
    4 CONTINUE
    5 A(I, I) = 1.0 / B
      RETURN
      END
```

*** SOURCE PROGRAM LISTING ***

```
      SUBROUTINECOE
      COMMON  GNU, RADO, RADN, NR, NP, PMU, WO, H,X1,Z1,X2,Z2,
     1S1, S2, S3, S4, F, FF, G, GG, SN, ST
      DIMENSION  X1(4),Z1(4),Z2(4),X2(4), S1(4,51 ),S2(4,51 ),S3(4,51),
     1S4(4,51 ), F(51),  FF(51), G(51), GG(51), SN(51), ST(51)
     1,TRX(4,4), A(4)
      NRR =(NR/NP)+1
      DD = GNU/(1.0 - GNU)
      DO 10 I = 1,4
      TRX(1,I) = S1(I,1) +DD * (S2(I,1)/RADO + 0.5*S4(I,1))
      TRX(2,I) = S1(I,NRR) + DD * (S2(I,NRR)/RADN + 0.5* S4(I,NRR))
      TRX(3,I) = (2.0/7.0)*S3(I,1) - S2(I,1)
      TRX(4,I) = (2.0/7.0)*S3(I,NRR) - S2(I,NRR)
   10 CONTINUE
      CALL INV(TRX)
      DO 20 K =1,4
      A(K)=-WO*(TRX(K,1) + TRX(K,2))*1.25*DD
   20 CONTINUE
      DO 30 IJ = 1, NRR
      F(IJ) = A(1)*S2(1,IJ)+A(2)*S2(2,IJ)+A(3)*S2(3,IJ)+A(4)*S2(4,IJ)
      G(IJ) = A(1)*S4(1,IJ)+A(2)*S4(2,IJ)+A(3)*S4(3,IJ)+A(4)*S4(4,IJ)
      FF(IJ) =A(1)*S1(1,IJ)+A(2)*S1(2,IJ)+A(3)*S1(3,IJ)+A(4)*S1(4,IJ)
      GG(IJ) =A(1)*S3(1,IJ)+A(2)*S3(2,IJ)+A(3)*S3(3,IJ)+A(4)*S3(4,IJ)
   30 CONTINUE
      RETURN
      END
```

*** S O U R C E   P R O G R A M   L I S T I N G ***

```
      SUBROUTINESTRESS
      COMMON  GNU, RADO, RADN, NR, NP, PMU, WO, H,X1,Z1,X2,Z2,
     1S1, S2, S3, S4, F, FF, G, GG, SN, ST
      DIMENSION  X1(4),Z1(4),Z2(4),X2(4), S1(4,51 ),S2(4,51 ),S3(4,51),
     1S4(4,51 ), F(51),  FF(51), G(51), GG(51), SN(51), ST(51)
      EE =(1.0 - GNU)/(1.0 - 2.0*GNU)
      NRR =(NR/NP)+1
      DO 10  IJ = 1,NRR
      SN(IJ) = 2.0 * PMU * EE * (WO - 2.0 * G(IJ))
      ST(IJ) = -2.0 * PMU * F(IJ)
   10 CONTINUE
      RETURN
      END
```

APPENDIX D

PURE TORSION IN LARGE-DEFORMATION ELASTICITY OF AN INCOMPRESSIBLE ELASTOMER

Take as a reference frame cylindrical coordinates r, θ, z in the deformed body. Then a point r, θ, z was initially at the point $\rho, \theta-\zeta z, z$, where ρ is assumed to be a function of r only, and the point of the undeformed body is given by

$$x_1 = \rho \cos(\theta-\zeta z),$$

$$x_2 = \rho \sin(\theta-\zeta z),$$

$$x_3 = z, \tag{D-1}$$

where ζ is the angle of rotation in the z direction.

The components G_{ij}, G^{ij} of the metric tensor of the deformed body are

$$G_{ij} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{Bmatrix}, \qquad G^{ij} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{Bmatrix}. \qquad \text{(D-2) \& (D-3)}$$

The components g_{ij}, g^{ij} of the metric tensor of the undeformed body are

$$g_{ij} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & r^2 & -\zeta r^2 \\ 0 & -\zeta r^2 & 1+\zeta^2 r^2 \end{Bmatrix}, \qquad g^{ij} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 + 1/r^2 & \zeta \\ 0 & \zeta & 1 \end{Bmatrix}. \qquad \text{(D-4) \& (D-5)}$$

Since $G = g = r^2$, the incompressibility equation $I_3 = 1$ is satisfied.

Consider the Mooney stress-strain relation for which the strain energy functions are given in terms of two constants, C_1 and C_2.

$$W = C_1(I_1 - 3) + C_2(I_2 - 3). \tag{D-6}$$

Thus, from equations (E-7) and (E-7a),

$$\Phi = 2C_1, \qquad \bar{\Psi} = 2C_2$$

$$Q_{ik} = \left\{ \begin{array}{ccc} 2+\zeta^2 r^2 & 0 & 0 \\ 0 & \zeta^2 + \frac{2}{r^2} & \zeta \\ 0 & \zeta & 2 \end{array} \right\}. \tag{D-7}$$

The contravariant stress tensor is

$$\left.\begin{aligned} \tau^{11} &= 2(C_1 + 2C_2) + 2C_2\, \zeta^2 r^2 + P \\ r^2 \tau^{22} &= 2(C_1 + 2C_2) + 2(C_1 + C_2)\, \zeta^2 r^2 + P \\ \tau^{33} &= 2(C_1 + 2C_2) + P \\ \tau^{33} &= 2(C_1 + C_2)\zeta \\ \tau^{31} &= \tau^{12} = 0 \end{aligned}\right\}. \tag{D-8}$$

The equilibrium equations are

$$\tau^{ik}_{\ \ i} + \Gamma^{i}_{ir}\, \tau^{rk} + \Gamma^{k}_{ir}\, \tau^{ir} = 0. \tag{D-9}$$

From the metric tensor for a deformed body,

$$\begin{aligned} \Gamma^1_{22} &= -r \\ \Gamma^2_{12} &= \Gamma^2_{21} = 1/r. \end{aligned} \tag{D-10}$$

The rest are zero.

The equilibrium equations are

$$\left.\begin{aligned} &\frac{\partial \tau^{11}}{\partial r} + \frac{1}{r}\tau^{11} - r\tau^{22} = 0 \\ &\frac{\partial \tau^{22}}{\partial \theta} = 0 \\ &\frac{\partial \tau^{33}}{\partial z} = 0 \end{aligned}\right\} \tag{D-11}$$

or

$$\left.\begin{aligned} &\frac{\partial P}{\partial r} + 4C_2\zeta^2 r + \frac{1}{r}\{2C_2\zeta^2 r^2 - 2(C_1 + C_2)\zeta^2 r^2\} = 0 \\ &\frac{\partial P}{\partial \theta} = 0 \\ &\frac{\partial P}{\partial z} = z \end{aligned}\right\} . \tag{D-12}$$

From equation (D-7),

$$\frac{\partial P}{\partial r} + (4C_2 - 2C_1)\zeta^2 r = 0$$

and

$$P = (C_1 - 2C_2)\zeta^2 r^2 + \alpha . \tag{D-13}$$

Since

$$\tau^{11} = 0 \quad \text{at } r = r_o \quad \text{(outer radius)}$$

$$P = -2(C_1 + 2C_2) - 2C_2\zeta^2 r_o^{\ 2}$$

$$\alpha = -2(C_1 + 2C_2) - C_1\zeta^2 r_o^{\ 2} \tag{D-14}$$

$$\therefore\ P = -2(C_1 + 2C_2) - C_1\zeta^2 r_o^{\ 2} + (C_1 - 2C_2)\zeta^2 r^2 \tag{D-15}$$

$$\left.\begin{aligned}
\tau^{11} &= -C_1\zeta^2 r_o^{\ 2} + C_1\zeta^2 r^2 = C_1\zeta^2(r^2 - r_o^{\ 2}) \\
r^2\tau^{22} &= 3C_1\zeta^2 r^2 - C_1\zeta^2 r_o^{\ 2} = C_1\zeta^2(3r^2 - r_o^{\ 2}) \\
\tau^{33} &= -C_1\zeta^2 r_o^{\ 2} + (C_1 - 2C_2)\zeta^2 r^2 \\
\tau^{23} &= 2(C_1 + C_2)\zeta
\end{aligned}\right\} . \qquad \text{(D-16)}$$

The physical components of stress are

$$\sigma_{ij} = \sqrt{G_{ij}/G^{ii}}\ \tau^{ij} \quad \text{(no sum)} \qquad \text{(D-17)}$$

$$\left.\begin{aligned}
\sigma_{11} &= \tau^{11} = C_1\zeta^2(r^2 - r_o^{\ 2}) \\
\sigma_{22} &= r^2\tau^{22} = C_1\zeta^2(3r^2 - r_o^{\ 2}) \\
\sigma_{33} &= \tau^{33} = C_1\zeta^2(r^2 - r_o^{\ 2}) - 2C_2\zeta^2 r^2 \\
\sigma_{23} &= r\tau^{23} = 2(C_1 + C_2)r\zeta
\end{aligned}\right\} . \qquad \text{(D-18)}$$

$$\sigma_{31} = \sigma_{12} = 0$$

The surface tractions at plane end (bonded surface) are

$$\left.\begin{aligned}
T_r &= 0 \\
T_\theta &= r\tau^{22} \\
T_z &= \tau^{33}
\end{aligned}\right\} . \qquad \text{(D-19)}$$

The moment required for the twist is

$$M = \int (rT\theta) r dr d\theta = \int r^3 \tau^{22} dr d\theta$$

$$= 2\pi \int_b^a C_1 \zeta^2 (3r^3 - r_o^2) dr$$

$$= 2\pi C_1 \zeta^2 \left[\frac{3r^4}{4} - \frac{r_o^2 r^2}{2}\right]_{r_i}^{r_o} = 2\pi C_1 \zeta^2 \left[+ \frac{r_o^4}{4} - \frac{r_i^4}{4} + \frac{r_o^2 r_i^2}{2}\right]$$

$$M = \frac{\pi}{2} C_1 \zeta^2 \left[r_o^4 - r_i^4 + 2 r_o^2 r_i^2\right] . \tag{D-20}$$

Force is also required to prevent elongation:

$$N = 2\pi \int_a^b \tau^{33} r dr = 2\pi \int_a^b \left[C_1 \zeta^2 (r^2 - r_o^2) - 2C_2 \zeta^2 r^2\right] r dr$$

$$= 2\pi\{-C_1 \zeta^2 r_o^2/2 \, (r_o^2 - r_i^2) + (C_1 - 2C_2)\zeta^2 \tfrac{1}{4}(r_o^4 - r_i^4)\}$$

$$= 2\pi\zeta^2\{- \frac{C_1}{4} r_o^4 + \frac{C_1}{2} r_o^2 r_i^2 - \frac{C_2}{2} (r_o^4 - r_i^4)\} . \tag{D-21}$$

It is found that in order to maintain this state of deformation, pressure must be applied at the inner curve surface. The magnitude of the pressure is

$$-C_1 \zeta^2 (\Gamma_o^2 - \Gamma_i^2)$$

where

Γ_i = inner radius

Γ_o = outer radius.

APPENDIX E

LARGE-DEFORMATION ELASTICITY THEORY FOR AXIAL COMPRESSION

In order to characterize the deformation of an elastic body, the position of each point in the body before and after deformation must be described. That is, if a point in the undeformed body is at y^1, y^2, y^3, then in the deformed body it will occupy the point x^1, x^2, x^3. For the current problem, it is convenient to employ cylindrical coordinates for this description; therefore, an undeformed system (r, θ', z') is defined by

$$y^1 = r \cos\theta', \qquad y^2 = r \sin\theta', \qquad y^3 = z \tag{E-1}$$

and a deformed system (ρ, θ, z) by

$$x^1 = \rho \cos\theta, \qquad x^2 = \rho \sin\theta, \qquad x^2 = z. \tag{E-2}$$

In particular, since the loading in the present problem is axially symmetric, the deformation may be considered in the form

$$\rho = \rho(r,z'), \qquad z = z(r,z') \text{ and } \theta = \theta' + t(r,z'). \tag{E-3}$$

In order to determine these functions, it is useful to consider another description of the deformation. For this purpose, the distortion which the initial polar coordinate system undergoes as the body deforms can be taken into account if these coordinates are considered to be embedded in the material. To characterize this distortion, the metric tensors G_{ij} and g_{ij}, which are associated with the polar coordinates before and after deformation, are defined as

$$G_{ij} = \frac{\partial y^r}{\partial \varphi^i} \frac{\partial y^r}{\partial \varphi^j} \quad \text{and} \quad g_{ij} = \frac{\partial x^r}{\partial \varphi^i} \frac{\partial x^r}{\partial \varphi^j} \tag{E-4}$$

where $\varphi^1 = r$, $\varphi^2 = \theta'$, $\varphi^3 = z'$. Using equation (E-1), we obtain

$$G_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \tag{E-5}$$

and from equations (E-2) and (E-3),

$$
g_{ij} = \begin{pmatrix} \left[\left(\frac{\partial \rho}{\partial r}\right)^2 + \rho^2 \left(\frac{\partial t}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2\right] & \rho^2 \frac{\partial t}{\partial r} & \left[\frac{\partial \rho}{\partial r}\frac{\partial \rho}{\partial z'} + \rho^2 \frac{\partial t}{\partial r}\frac{\partial t}{\partial z'} + \frac{\partial}{\partial r}\frac{\partial z}{\partial z'}\right] \\ \rho^2 \frac{\partial t}{\partial r} & \rho^2 & \rho^2 \frac{\partial t}{\partial z'} \\ \left[\frac{\partial \rho}{\partial r}\frac{\partial \rho}{\partial z'} + \rho^2 \frac{\partial t}{\partial r}\frac{\partial t}{\partial z} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'}\right] & \rho^2 \frac{\partial t}{\partial z'} & \left[\left(\frac{\partial z}{\partial z'}\right)^2 + \rho^2 \left(\frac{\partial t}{\partial z'}\right) + \left(\frac{\partial \rho}{\partial z'}\right)^2\right] \end{pmatrix}. \qquad \text{(E-6)}
$$

In terms of the above metrics, G_{ij} and g_{ij}, the stresses τ^{ij} in the body may now be determined. As a first approach to the solution of the problem under consideration, we will consider the material to be incompressible and use constitutive relations given by

$$\tau^{ij} = \Phi G^{ij} + \Psi Q^{ij} + \quad g_{ij} \qquad \text{(E-7)*}$$

where

$$Q^{ij} = G^{rs} g_{rs} G^{ij} - G^{ir} G^{is} g_{rs} , \qquad \text{(E-7a)*}$$

Ψ and Φ are constants, and ρ is a scalar function of position. Furthermore, from equations (E-5) and (E-6), the contravariant components of the undeformed and deformed coordinates which appear in equation (E-7) are found to be

$$G^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(E-8)}$$

and

$$g^{ij} = \begin{pmatrix} \text{Matrix where each component} \\ \text{is the cofactor of the cor-} \\ \text{responding component in array} \\ \text{(E-6). This is shown later as} \\ \text{(E-21).} \end{pmatrix}. \qquad \text{(E-9)}$$

---

*See reference 2, equations 1.15.13 and 1.15.9, page 28.

Since the stresses are defined, it is now possible to obtain a set of equations whose solution will yield the desired functions (E-3). These equations are obtained by using the equilibrium equations

$$\tau^{ij}_{,i} + \Gamma^{i}_{ri}\,\tau^{rj} + \Gamma^{j}_{im}\,\tau^{im} = 0 \qquad \text{(E-10)}$$

where the Γ^{r}_{st} are the Christoffel symbols of the second kind defined by

$$\Gamma^{r}_{st} = \tfrac{1}{2}\, g^{kr}\,[g_{sk,t} + g_{tk,s} - g_{st,k}] \qquad \text{(E-11)}$$

and the stresses are given by equations (E-7).

It may be observed that, due to the complexity of the tensors g^{ij} and g_{ij}, the equilibrium equations as defined above will be extremely cumbersome (see Appendix D equations). It may also be observed that if we had, in relation (E-4), identified φ^{i}'s with the deformed rather than the undeformed coordinates, the resulting form of the equilibrium equations would have been greatly simplified. The following considerations, however, indicate that this approach is not practical, since it introduces difficulties in the boundary conditions. If the undeformed coordinates appear as the independent variables in the equilibrium equations, as is currently the case, the boundary is located by specifying its undeformed position. If, however, the equilibrium equations are simplified, as described above, the deformed coordinates will appear as the independent variable and the boundary must be located by specifying its deformed position, which is unknown until the problem has been solved. It is observed that the three equations (E-10) contain four unknowns: ρ, t, z, and P. However, using the fact that the material is incompressible, a fourth equation is obtained,

$$\frac{|G_{ij}|}{|g_{ij}|} = 1\,, \qquad \text{(E-12)}$$

which states that the volume of any portion of the body remains constant.

Up to this point, the loading history of the body has not been discussed. However, it is apparent that unlike in a linear elastic theory, the order of loading must be specified. In order to build this information into the equations, the following considerations are used: Let ρ, t, z, and P be made up of contributions due to compression and torsion (i. e., let $\rho = \rho_c + \rho_t$, $z = z_c + z_t$, etc., where ρ_c and ρ_t represent components

after compression and tension, respectively). Since the compression is applied first, equations (E-10) are solved for ρ_c, z_c, and ρ_c (i.e., $t_c = 0$) with $\rho_t = t_c = z_t = P_t = 0$. Having solved for the compression contribution, the torsion contribution is now determined. It is observed that the torsion considerations are influenced by the initial compression by way of the compression-torsion coupling terms which appear in the equilibrium equations.

Having set up the field equations, we must now consider the boundary conditions.

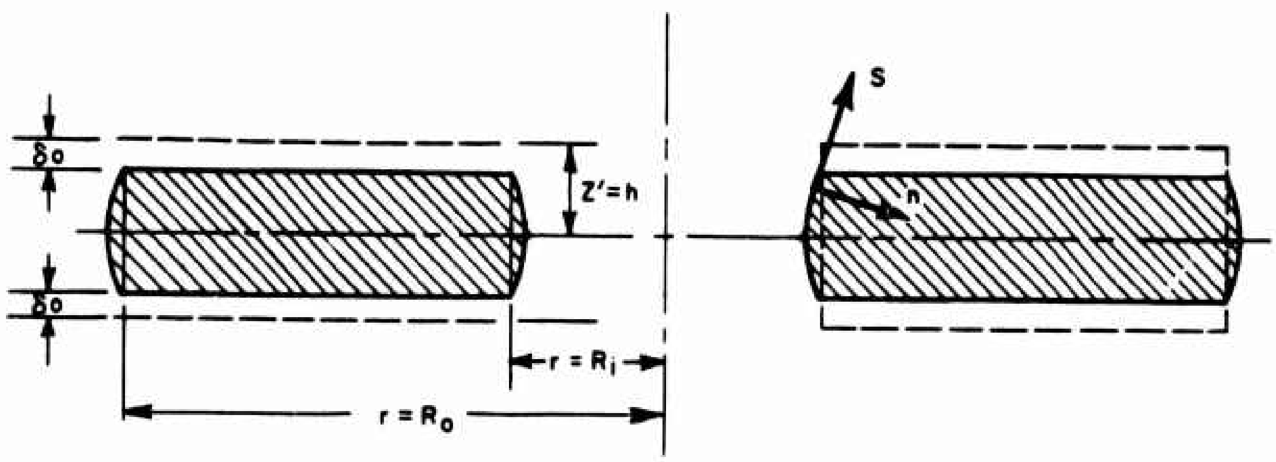


Figure 15. Sketch of Coordinates for One Lamination.

For the compression problem, the boundary conditions which must be satisfied are as follows:

at

$$z' = \pm h, \qquad z_c = \pm (h - \delta_o), \qquad \rho_c = r \tag{E-13}$$

at $r = R_o$, R_i, traction is zero, or in terms of a coordinate system on the boundary, $\tau^{nn} = \tau^{ns} = 0$,

or

$$\tau_c^{nn} = \frac{\left(\frac{d\rho}{dz}\right)^2 \tau_c^{rr} + 2\left(\frac{d\rho}{dz}\right) \tau_c^{rz'} + \tau_c^{z'z'}}{1 + \left(\frac{d\rho}{dz}\right)^2} = 0$$

$$= \quad \tau_i^{ns} = \frac{-\left(\frac{d\rho}{dz}\right) \tau_c^{rr} + \left[\left(\frac{d\rho}{dz}\right)^2 - 1\right] \tau_c^{rz'} + \left(\frac{d\rho}{dz}\right) \tau_c^{z'z'}}{1 + \left(\frac{d\rho}{dz}\right)^2} = 0. \tag{E-14}$$

And for the torsion problem,

at $z' = \pm h, \quad z_t = z_c \pm (h - \delta_o), \quad \rho_t = \rho_c = r$ (specified twist angle)

$\tau_t = \tau_o$

at $r = R_o, R_i, \quad \tau_t^{nn} = \tau_c^{nn} = 0, \ \tau_t^{ns} = \tau_c^{ns} = 0$ and $\tau_t^{\theta'\theta'} = 0.$

Having formulated the problem, a method for obtaining the solution will now be considered. For this purpose, consider rewriting the equilibrium equations and boundary conditions using the displacements

$$\rho_c - r = u_c, \qquad \rho_t - r = u_t$$

$$z_c - z' = w_c, \qquad z_t - z' = w_t, \text{ etc.,}$$

rather than the deformed positions, as dependent variables.

If it is considered now that the linear elastic equations are given in the form

$$f(u, w, P) = 0, \ g(u,w,p) = 0, \quad h(u,w) = 0,$$

then, with the above change of variables, the nonlinear equations will be

$$f(u,w,P) = F(u,w,P), \qquad g(u,w,P) = G(u,w,P)$$

and

$$h(u,w) = H(u,w)$$

where F, G, and H represent nonlinear contributions. The solution to these equations can now be obtained by using a numerical technique similar to that used in the solution of the linear equations. To do this, the nonlinear terms, F, G, H, which appear in the equations are treated as if they represent a nonhomogeneous portion of the linear elastic equations. That is, in each iteration it is considered that the nonlinear terms are known, the value being given by the value obtained from the previous iteration.

Combined Compression and Torsion

Assume that a rubber annulus is placed under axisymmetric compression followed by torsion. Consider deformation described by

deformed coordinates	(ρ, θ, z)
undeformed coordinates	(r, θ', z')
where	$\rho = \rho(r, z')$
	$z = z(r, z')$
	$\theta = \theta' + t(r, z')$.

The metric of φ in the undeformed coordinates is

$$G_{ij} = \frac{\partial y^r}{\partial \varphi^i} \frac{\partial y^r}{\partial \varphi^j} \tag{E-15}$$

where

$$y^1 = r \cos\theta'$$
$$y^2 = r \sin\theta'$$
$$y^3 = z'$$

and

$$\varphi^1 = r$$
$$\varphi^2 = \theta'$$
$$\varphi^3 = z';$$

then

$$G_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{E-16}$$

$$|G_{ij}| = r^2 \tag{E-16a}$$

and

$$G^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{E-17}$$

$$|G^{ij}| = 1/r^2. \tag{E-17a}$$

The metric of φ in the deformed coordinates is

$$g_{ij} = \frac{\partial x^r}{\partial \varphi^i} \frac{\partial x^r}{\partial \varphi^j} \tag{E-18}$$

where

$$x^1 = \rho \cos\theta$$
$$x^2 = \rho \sin\theta$$
$$x^3 = z \, .$$

Expanding the separate derivatives in equation (E-18) yields

$$\frac{\partial x^1}{\partial \varphi^1} = \frac{\partial x^1}{\partial r} = \frac{\partial x^1}{\partial \rho}\frac{\partial \rho}{\partial r} + \frac{\partial x^1}{\partial \theta}\frac{\partial \theta}{\partial r} = \cos\theta \frac{\partial \rho}{\partial r} - \rho \sin\theta \frac{\partial t}{\partial r}$$

$$\frac{\partial x^2}{\partial \varphi^1} = \frac{\partial x^2}{\partial r} = \frac{\partial x^2}{\partial \rho}\frac{\partial \rho}{\partial r} + \frac{\partial x^1}{\partial \theta}\frac{\partial \theta}{\partial r} = \sin\theta \frac{\partial \rho}{\partial r} + \rho \cos\theta \frac{\partial t}{\partial r}$$

$$\frac{\partial x^3}{\partial \varphi^1} = \frac{\partial x^3}{\partial r} = \frac{\partial x^3}{\partial z}\frac{\partial z}{\partial r} = \frac{\partial z}{\partial r}$$

$$\frac{\partial x^1}{\partial \varphi^2} = \frac{\partial x^1}{\partial \theta} = \frac{\partial x^1}{\partial \theta}\frac{\partial \theta}{\partial \theta'} = - \rho \sin\theta$$

$$\frac{\partial x^2}{\partial \varphi^2} = \frac{\partial x^2}{\partial \theta'} = \frac{\partial x^2}{\partial \theta}\frac{\partial \theta}{\partial \theta'} = \rho \cos\theta$$

$$\frac{\partial x^3}{\partial \varphi^2} = \frac{\partial x^3}{\partial \theta'} = 0$$

$$\frac{\partial x^1}{\partial \varphi^3} = \frac{\partial x^1}{\partial z'} = \frac{\partial x^1}{\partial \rho}\frac{\partial \rho}{\partial z'} + \frac{\partial x^1}{\partial \theta}\frac{\partial \theta}{\partial z'} = \cos\theta \frac{\partial \rho}{\partial z'} - \rho \sin\theta \frac{\partial t}{\partial z'}$$

$$\frac{\partial x^2}{\partial \varphi^3} = \frac{\partial x^2}{\partial z'} = \frac{\partial x^2}{\partial \rho}\frac{\partial \rho}{\partial z'} + \frac{\partial x^2}{\partial \theta}\frac{\partial \theta}{\partial z'} = \sin\theta \frac{\partial \rho}{\partial z'} + \rho \cos\theta \frac{\partial t}{\partial z'}$$

$$\frac{\partial x^3}{\partial \varphi^3} = \frac{\partial x^3}{\partial z'} = \frac{\partial x^3}{\partial z}\frac{\partial z}{\partial z'} = \frac{\partial z}{\partial z'} ;$$

then

$$g_{ij} = \begin{pmatrix} [(\frac{\partial\rho}{\partial r})^2 + \rho^2 (\frac{\partial t}{\partial r})^2 + (\frac{\partial z}{\partial r})^2] & \rho^2 \frac{\partial t}{\partial r} & [\frac{\partial\rho}{\partial r}\frac{\partial\rho}{\partial z'} + \rho^2 \frac{\partial t}{\partial r}\frac{\partial t}{\partial z'} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'}] \\ \rho^2 \frac{\partial t}{\partial r} & \rho^2 & \rho^2 \frac{\partial t}{\partial z'} \\ [\frac{\partial\rho}{\partial r}\frac{\partial\rho}{\partial z'} + \rho^2 \frac{\partial t}{\partial r}\frac{\partial t}{\partial z'} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'}] & \rho^2 \frac{\partial t}{\partial z'} & [(\frac{\partial z}{\partial z'})^2 + \rho^2 (\frac{\partial t}{\partial z'})^2 + (\frac{\partial \rho}{\partial z'})^2] \end{pmatrix}, \tag{E-19}$$

which implies

$$\begin{aligned} |g_{ij}| &= -\rho^2 t_r \{\rho^2 t_r [Z^2_{z'} + \rho^2 t^2_{z'} + \rho^2_{z'}] - \rho^2 t_{z'} [\rho_r \rho_{z'} + \rho^2 t_r t_{z'} + Z_r Z_{z'}]\} \\ &+ \rho^2 \{[\rho^2_r + \rho^2 t^2_r + Z^2_r][Z^2_r + \rho^2 t^2_{z'} + \rho^2_{z'}] - [\rho_r \rho_{z'} + \rho^2 t_r t_{z'} + Z_r Z_{z'}][\rho_r \rho_{z'} \\ &+ \rho^2 t_r t_{z'} + Z_r Z_{z'}]\} - \rho^2 t_{z'} \{\rho^2 t_{z'} [\rho^2_r + \rho^2 t^2_r + Z^2_r] \\ &- \rho^2 t_r [\rho_r \rho_{z'} + \rho^2 t_r t_{z'} + Z_r Z_{z'}]\} \\ &= \rho^2 [\rho^2_r Z^2_{z'} + Z^2_r Z^2_{z'} + \rho^2_r \rho^2_{z'} + Z_r \rho^2_{z'} - \rho^2_r \rho^2_{z'} - Z^2_r Z^2_{z'} - 2\rho_r \rho_{z'} Z_r Z_{z'}] \end{aligned}$$

and

$$|g_{ij}| = \rho^2 [\frac{\partial\rho}{\partial r}\frac{\partial z}{\partial z'} - \frac{\partial z}{\partial r}\frac{\partial \rho}{\partial z'}]^2 \equiv [\rho h(r,z')]^2. \tag{E-20}$$

Thus,

$$g_{ij} = \frac{1}{[\rho h]^2} \begin{pmatrix} \rho^2 [(\frac{\partial z}{\partial z'})^2 + (\frac{\partial \rho}{\partial z'})^2] & \rho^2 \{\frac{\partial t}{\partial z'} [\frac{\partial \rho}{\partial r} \frac{\partial \rho}{\partial z'} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial z'}] + \frac{\partial t}{\partial r} [(\frac{\partial z}{\partial z'})^2 + (\frac{\partial \rho}{\partial z'})^2]\} & -\rho^2 [\frac{\partial \rho}{\partial r} \frac{\partial \rho}{\partial z'} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial z'}] \\ \rho^2 \{\frac{\partial t}{\partial z'} [\frac{\partial \rho}{\partial r} \frac{\partial \rho}{\partial z'} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial z'}] + \frac{\partial t}{\partial r} [(\frac{\partial z}{\partial z'})^2 + (\frac{\partial \rho}{\rho z'})^2]\} & \rho^2 [(\frac{\partial \rho}{\partial r} \frac{\partial t}{\partial z'} - \frac{\partial t}{\partial r} \frac{\partial \rho}{\partial z'})^2 + (\frac{\partial t}{\partial r} \frac{\partial z}{\partial z'} - \frac{\partial t}{\partial z'} \frac{\partial z}{\partial r})^2] + (\frac{\partial \rho}{\partial r} \frac{\partial z}{\partial z'} - \frac{\partial t}{\partial r} \frac{\partial \rho}{\partial z'}) & \rho^2 \{\frac{\partial t}{\partial z'} [(\frac{\partial \rho}{\partial r})^2 + (\frac{\partial z}{\partial r})^2] + \frac{\partial t}{\partial r} [\frac{\partial \rho}{\partial r} \frac{\partial \rho}{\partial z'} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial z'}]\} \\ -\rho^2 [\frac{\partial \rho}{\partial r} \frac{\partial \rho}{\partial z'} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial z'}] & \rho^2 \{\frac{\partial t}{\partial z'} [(\frac{\partial \rho}{\partial r})^2 + (\frac{\partial z}{\partial r})^2] + \frac{\partial t}{\partial r} [\frac{\partial \rho}{\partial r} \frac{\partial \rho}{\partial z'} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial z'}]\} & \rho^2 [(\frac{\partial \rho}{\partial r})^2 + (\frac{\partial z}{\partial r})^2] \end{pmatrix}. \quad \text{(E-21)}$$

Now the strain invariants are

$$I_1 = G^{ij} g_{ij} = g_{11} + \frac{1}{r^2} g_{22} + g_{33} \quad \text{(E-22)}$$

and

$$I_3 = \frac{|G_{ij}|}{|g_{ij}|} = \frac{r^2}{\rho^2 h^2}, \quad \text{(E-23)}$$

but for incompressibility, $I_3 = 1$; hence,

$$\rho = \frac{r}{h(r,z')}. \quad \text{(E-24)}$$

Also, from equation (E-7a),

$$Q^{ij} = I_1 \, G^{ij} - G^{ir} \, G^{js} \, g_{rs}$$

where G^{ij} is the diagonal $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Thus, the components of G^{ij} are

$$\left.\begin{aligned}
Q^{11} &= I_1 \, G^{11} - G^{11} \, G^{11} \, g_{11} = 1/r^2 \, g_{22} + g_{33} \\
Q^{22} &= I_1 \, G^{22} - G^{22} \, G^{22} \, g_{22} = 1/r^2 \, (g_{11} + g_{33}) \\
Q^{33} &= I_1 \, G^{33} - G^{33} \, G^{33} \, g_{33} = g_{11} + 1/r^2 \, g_{22} \\
Q^{12} &= Q^{21} = G^{11} \, G^{22} \, g_{12} = - 1/r^2 \, g_{12} \\
Q^{13} &= Q^{31} = - G^{11} \, G^{33} \, g_{13} = - g_{13} \\
Q^{23} &= Q^{32} = - G^{22} \, G^{33} \, g_{23} = - 1/r^2 \, g_{23}
\end{aligned}\right\} \cdot \qquad \text{(E-25)}$$

Thus,

$$\underset{\substack{\text{deformed} \\ \text{coordinates}}}{\tau^{ij}} = \Phi \, \underset{\substack{\text{undeformed} \\ \text{coordinates}}}{G^{ij}} + \Psi \, Q^{ij} + P \, \underset{\substack{\text{deformed} \\ \text{coordinates}}}{g^{ij}}$$

which is then substituted into the equilibrium equations, (E-10).

Compression Only

For the problem of axial compression only, the formulation is carried further as follows:

For the case of notation, let

$$\left.\begin{aligned} u_c &= u \\ z &= z' + \\ \rho &= r + \end{aligned}\right\} . \qquad \text{(E-26)}$$

From equation (E-17),

$$G^{ij} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

From equation (E-20),

$$h = \left(\frac{\partial\rho}{\partial r}\frac{\partial z}{\partial z'} - \frac{\partial z}{\partial r}\frac{\partial\rho}{\partial z'}\right) = \left[\left(1 + \frac{\partial u}{\partial r}\right)\left(1 + \frac{\partial w}{\partial z'}\right) - \frac{\partial w}{\partial r}\frac{\partial u}{\partial z'}\right]; \qquad \text{(E-27)}$$

and from equation (E-23), h is also given by

$$h = r/\rho \ . \qquad \text{(E-27a)}$$

From equation (E-21), the components of g^{ij} are

$$\left.\begin{aligned}
g^{11} &= \frac{1}{h^2}\left[\left(\frac{\partial z}{\partial z'}\right)^2 + \left(\frac{\partial\rho}{\partial z'}\right)^2\right] = \frac{1}{h^2}\left[\left(1 + \frac{\partial w}{\partial z'}\right)^2 + \left(\frac{\partial w}{\partial z'}\right)^2\right] \\
g^{22} &= \frac{1}{\rho^2 h^2}\left[\frac{\partial\rho}{\partial r}\frac{\partial z}{\partial z'} - \frac{\partial z}{\partial r}\frac{\partial\rho}{\partial z'}\right]^2 = \frac{1}{\rho^2} = \frac{1}{r^2\left(1+\frac{u}{r}\right)^2} = \frac{h^2}{r^2} \\
g^{33} &= \frac{1}{h^2}\left[\left(\frac{\partial\rho}{\partial r}\right)^2 + \left(\frac{\partial t}{\partial r}\right)^2\right] = \frac{1}{h^2}\left[\left(1 + \frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2\right] \\
g^{13} &= g^{31} = \frac{1}{h^2}\left[\frac{\partial\rho}{\partial r}\frac{\partial\rho}{\partial z'} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'}\right] = -\frac{1}{h^2}\left[\left(1 + \frac{\partial u}{\partial r}\right)\frac{\partial u}{\partial z'} + \frac{\partial w}{\partial r}\left(1 + \frac{\partial w}{\partial z'}\right)\right] \\
g^{12} &= g^{21} = g^{23} = g^{32} = 0 \\
&\qquad \text{(Since torsion is zero, } \quad = 0)
\end{aligned}\right\} .$$

(E-28)

From equation (E-19), the components of g_{ij} are

$$
\left.
\begin{aligned}
g_{11} &= \left(\frac{\partial\rho}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = \left(1 + \frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2 = h^2 g^{33} \\
g_{22} &= \rho^2 = (u + r)^2 = r^2(1 + u/r)^2 = r^2/h^2 \\
g_{33} &= \left(\frac{\partial z}{\partial z'}\right)^2 + \left(\frac{\partial\rho}{\partial z'}\right)^2 = \left(1 + \frac{\partial w}{\partial z'}\right)^2 + \left(\frac{\partial u}{\partial z'}\right)^2 = h^2 g^{11} \\
g_{31} &= g_{13} = \frac{\partial\rho}{\partial r}\frac{\partial\rho}{\partial z'} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z'} = \left(1 + \frac{\partial u}{\partial r}\right)\frac{\partial u}{\partial z'} + \frac{\partial w}{\partial z'}\left(1 + \frac{\partial w}{\partial z'}\right) = -h^2 g^{31} = h^2 g^{13} \\
g_{23} &= g_{32} = g_{21} = g_{12} \equiv 0
\end{aligned}
\right\}. \tag{E-29}
$$

Now the components of τ^{ij} from equation (E-7), by substituting from equations (E-17), (E-25), (E-28), and (E-29), are

$$
\left.
\begin{aligned}
\tau^{11} &= \Phi + \frac{\Psi}{h^2} + \left[\left(1 + \frac{\partial W}{\partial z}\right)^2 + \left(\frac{\partial U}{\partial z'}\right)^2\right]\left(\Psi + \frac{P}{h^2}\right) \\
\tau^{22} &= \frac{\Phi}{r^2} + \frac{\Psi}{r^2}(g_{11} + g_{33}) + P\frac{h^2}{r^2} \\
&= \frac{1}{r^2}\left\{\Phi + \Psi\left[\left(1 + \frac{\partial U}{\partial r}\right)^2 + \left(1 + \frac{\partial W}{\partial z'}\right)^2 + \left(\frac{\partial W}{\partial r}\right)^2 + \left(\frac{\partial U}{\partial z'}\right)^2\right] + Ph^2\right\} \\
\tau^{33} &= \Phi + \Psi\left(g_{11} + 1/r^2\, g_{22}\right) + P\frac{g_{11}}{h^2} = \Phi + \frac{\Psi}{h^2} \\
&\quad + \left[\left(1 + \frac{\partial U}{\partial r}\right)^2 + \left(\frac{\partial W}{\partial r}\right)^2\right]\left(\Psi + P/h^2\right) \\
\tau^{13} &= \tau^{31} = \left(\Psi + \frac{P}{h^2}\right)\left[\left(1 + \frac{\partial U}{\partial r}\right)\left(\frac{\partial U}{\partial z'}\right) + \frac{\partial W}{\partial r}\left(1 + \frac{\partial W}{\partial z'}\right)\right] \\
\tau^{23} &= \tau^{32} = \tau^{12} = \tau^{21} \equiv 0
\end{aligned}
\right\}. \tag{E-30}
$$

The equilibrium equations are now

$$\tau^{ij}/_{i} = 0 \tag{E-31}$$

or

$$\tau^{ij}_{,i} + \Gamma^{i}_{ri}\,\tau^{rj} + \Gamma^{j}_{im}\,\tau^{im} = 0\ ,$$

as shown in Equation (E-10),

where

$$\Gamma^{r}_{st} = \tfrac{1}{2}\, g^{kr}\, [g_{sk,t} + g_{tk,s} - g_{st,k}]\ . \tag{E-32}$$

Note: $\Gamma^{r}_{st} = \Gamma^{r}_{ts}$.

In the current problem of compression only, the only nonzero Christoffel symbols are

$$\left.\begin{aligned}
\Gamma^{1}_{11} &= \tfrac{1}{2}\,[g^{11}\, g_{11,1} + g^{13}\,(2g_{13,1} - g_{11,3})] \\
\Gamma^{1}_{22} &= -\tfrac{1}{2}\,(g^{11}\, g_{22,1} + g^{13}\, g_{22,3}) \\
\Gamma^{1}_{13} &= \Gamma^{1}_{31} = \tfrac{1}{2}\,(g^{11}\, g_{11,3} + g^{13}\, g_{33,1}) \\
\Gamma^{1}_{33} &= \tfrac{1}{2}\,[g^{13}\, g_{33,3} + g^{11}\,(2g_{13,3} - g_{33,1}) \\
\Gamma^{2}_{12} &= \Gamma^{2}_{21} = \tfrac{1}{2} g^{22}\, g_{22,1} \\
\Gamma^{2}_{23} &= \Gamma^{2}_{32} = \tfrac{1}{2} g^{22}\, g_{22,3} \\
\Gamma^{3}_{11} &= \tfrac{1}{2}\,[g^{13}\, g_{11,1} + g^{33}\,(2g_{13,1} - g_{11,3})] \\
\Gamma^{3}_{22} &= -\tfrac{1}{2}\,[g^{31}\, g_{22,1} + g^{33}\, g_{22,3}] \\
\Gamma^{3}_{33} &= \tfrac{1}{2}\,[\, g^{33}\, g_{33,3} + g^{31}\,(2g_{13,3} - g_{33,1})] \\
\Gamma^{3}_{31} &= \Gamma^{3}_{13} = \tfrac{1}{2}\,[g^{13}\, g_{11,3} + g^{33}\, g_{33,1}]
\end{aligned}\right\}\ . \tag{E-33}$$

Now the equilibrium equations become

$$\underline{\tau^{11}_{,1}} + \underline{\tau^{31}_{,2}} + 2\,\Gamma^1_{11}\tau^{11} + 3\,\Gamma^1_{31}\tau^{31} + \underline{\Gamma^2_{12}\tau^{11}} + \Gamma^2_{32}\tau^{31}$$

$$+ \Gamma^3_{33}\tau^{31} + \Gamma^3_{13}\tau^{11} + \Gamma^1_{33} + \underline{\Gamma^1_{22}\tau^{22}} = 0 \qquad \text{(E-34a)}$$

and

$$\underline{\tau^{33}_{,3}} + \underline{\tau^{13}_{,1}} + 2\,\Gamma^3_{33}\tau^{33} + 3\,\Gamma^3_{13}\tau^{13} + \Gamma^2_{32}\tau^{33} + \underline{\Gamma^2_{12}\tau^{13}}$$

$$+ \Gamma^1_{11}\tau^{13} + \Gamma^1_{31}\tau^{33} + \Gamma^3_{11}\tau^{11} + \Gamma^3_{22}\tau^{22} = 0\,. \qquad \text{(E-34b)}$$

(Note: The quantities underlined are those that contribute linear terms.)

The third equilibrium is satisfied identically, as expected, since there is no dependence upon angle θ (axisymmetric). Now consider rewriting the stress component in the form of linear terms plus the nonlinear contribution. To do this, consider first the continuity equation from equations (E-27) and (E-27a).

For convenience, the following notation will be used:

$$\frac{\partial u}{\partial r} = u_r, \quad \frac{\partial u}{\partial z'} = u_{z'}, \text{ etc.} \qquad \text{(E-35)}$$

From equations (E-27) and (E-27a)

$$(1 + u_r)\,(1 + w_{z'}) - w_r\, u_{z'} = \frac{1}{1 + \frac{u}{r}}$$

or

$$(1 + u_r)\,(1 + w_z)\,(1 + \tfrac{u}{r}) - w_r \;\; {}_{z'}\,(1 + \tfrac{u}{r}) = 1. \qquad \text{(E-36)}$$

Expanding, this yields

$$u_r + w_{z'} + \frac{u}{r} = -\,(u_r\, w_{z'} + \frac{u}{r} w_{z'} + \frac{u}{r} u_r + u_r\, w_{z'}\, \frac{u}{r} + w_r\, u_{z'}(1 + \frac{u}{r})$$

or

$$u_r + w_{z'} + \frac{u}{r} = A \qquad \text{(E-37)}$$

where $A = -\,u_r\, w_{z'} + \frac{u}{r} w_{z'} + \frac{u}{r} u_r + u_r\, w_{z'}\, \frac{u}{r} + w_r\, u_{z'}\,(1 + \frac{u}{r})$.

Let (r, z) be initial coordinates and

(r', z') be final coordinates.

Equilibrium equations are written with r and z as independent variables. A location is then described by its initial coordinates; one then solves for r' and z'. The stresses are expressed in the final coordinates; and on the traction-free boundary, the free inner and outer edge, the normal and shear stresses, σ_{nn} and σ_{ns}, are required to be zero. To express this in terms of $\sigma_{r'r'}$, $\sigma_{z'z'}$, and $\sigma_{z'r'}$, which are in turn functions of the coordinates, one needs the boundary shape. That is, the quantity $\frac{d\rho}{dz}$ is needed where the boundary location is given as r = R. If a finite-difference method is to be employed, the derivative, $\frac{d\rho}{dz}$, is equivalent to $\frac{\rho'_{i+1} - \rho'_{i-1}}{z'_{j+1} - z'_{j-1}}$.
With this approach, the grid shape is constant and the equilibrium equations are written in the deformed coordinates, which is the more convenient form.

Now, if we let $P = P' - (\Phi + 2\Psi)$, (E-38)

then expressions (E-30) yield

$$\tau^{11} = \Phi + \Psi\left(1 + \frac{u}{r}\right)^2 + (1 + 2w_{z'} + w_{z'}^2 + u_{z'}^2)\left[\Psi + (P' - \Phi - 2\Psi)\left(1 + \frac{u}{r}\right)^2\right]$$

$$\tau^{11} = \left(\Phi + \Psi + 2\Psi\frac{u}{r} + \left(\frac{u}{r}\right)^2\Psi\right) + \left[\Psi + P' - \Phi - 2\Psi + 2\frac{u}{r}P' - 2\frac{u}{r}(\Phi + 2\Psi)\right.$$
$$\left. + (P' - \Phi - 2\Psi)\left(\frac{u}{r}\right)^2\right] + 2w_{z'}\Psi + 2w_{z'}(u' - \Phi - 2\Psi) + 2w_{z'}(P' - \Phi - 2\Psi)\left[2\frac{u}{r} + \left(\frac{u}{r}\right)^2\right]$$
$$+ (w_{z'}^2 + u_{z'}^2)\left[\Psi + (P' - \Phi - 2\Psi)\left(1 + \frac{u}{r}\right)^2\right].$$

$$\tau^{11} = 2\Psi\frac{u}{r} - 2\frac{u}{r}(\Phi + 2\Psi) + 2w_{z'}(\Phi + 2\Psi) + P' + B \qquad \text{(E-39)}$$

where

$$B = \left(\frac{u}{r}\right)^2\Psi + 2\frac{u}{r}P' + (P' - \Phi - 2\Psi)\left(\frac{u}{r}\right)^2 + 2w_{z'}(P' - \Phi - 2\Psi)\left[2\frac{u}{r} + \left(\frac{u}{r}\right)^2\right]$$
$$+ (w_{z'}^2 + u_{z'}^2)\left[\Psi + (P' - \Phi - 2\Psi)\left(1 + \frac{u}{r}\right)^2\right].$$

Then

$$\tau^{11} = -2(\Phi + \Psi)\left(w_{z'} + \frac{u}{r}\right) + P' + B, \qquad \text{(E-40)}$$

and from continuity, equation (E-37),

$$\Phi - (w_{z'} + \frac{u}{r}) = u_r - A$$

or

$$\tau^{11} = \underbrace{2(\Phi + \Psi)\, u_r + P'}_{\substack{\text{linear terms can}\\ \text{identify Lame's}\\ \text{constant}\\ \mu=\Phi+\Psi}} - \underbrace{2(\Phi + \Psi)A + B}_{\substack{\text{nonlinear}\\ \text{contribution}}}; \tag{E-41}$$

also,

$$\tau^{13} = - [\Psi + (P' - \Phi - 2\Psi)(1 + \frac{u}{r})^2 \; [u_{z'} + w_r + u_r\, u_{z'} + w_r\, w_{z'}]$$

$$= - \{-(\Phi + \Psi)(u_{z'} + w_r) + [P'(1 + \frac{u}{r})^2 - (\Phi + 2\Psi)(2\,\frac{u}{r} + (\frac{u}{r})^2)]$$

$$[u_{z'} + w_r + u_r\, u_{z'} + w_r\, w_{z'}] + \Psi\, [u_r\, u_{z'} + w_r\, w_{z'}]\}$$

$$= \underbrace{(\Phi + \Psi)\, (u_{z'} + w_r)}_{\substack{\text{linear terms with}\\ \mu\, =\, \Phi + \Psi}} + \underbrace{C}_{\substack{\text{nonlinear}\\ \text{contribution}}} \tag{E-42}$$

where

$$C = -\{[P'(1+ \frac{u}{r})^2 - (\Phi+2\Psi)(2\,\frac{u}{r} + (\frac{u}{r})^2)]\; [u_{z'} + w_r + u_r\, u_{z'} + w_r\, {}_{z'}]$$

$$+ \Psi[u_r\, u_{z'} + w_r\, w_{z'}]\};$$

also,

$$\tau^{22} = \frac{1}{r^2\,(1 + \frac{u}{r})^2}\, \{\Phi\, (1 + \frac{u}{r})^2 + \Psi(1 + \frac{u}{r})^2\; [z + z\, u_r + 2w_{z'} + u_r^2 + w_{z'}^2$$

$$+ w_r^2 + u_{z'}^2] + P' - \Phi - 2\Psi\}$$

$$= \frac{1}{r^2\,(1 + \frac{u}{r})^2}\, \{2\,\frac{u}{r}\,(\Phi+2\Psi) + 2\Psi(u_r + w_{z'}) + P' + 2\Psi\, (\frac{u}{r})^2$$

$$+ \Psi[2\,\frac{u}{r} + (\frac{u}{r})^2][2u_r + 2w_{z'} + u_r^2 + w_{z'}^2 + w_r^2 + u_{z'}^2]\}$$

from continuity $u_r + w_{z'} = A - \frac{u}{r}$, or

$$\tau^{22} = \frac{1}{r^2 (1 + \frac{u}{r})^2} [\underbrace{2(\Phi + \Psi) \frac{u}{r} + P'}_{\text{linear}} + \underbrace{2 \Psi A + D}_{\text{nonlinear}}] \tag{E-43}$$

where

$$D = 2\Psi (\frac{u}{r})^2 + \Psi[2 \frac{u}{r} + (\frac{u}{r})^2][2u_r + 2w_{z'} + u_r^2 + w_{z'}^2 + w_r^2 + u_{z'}^2] .$$

It is seen here that the factor $\frac{1}{r^2 (1 + \frac{u}{r})^2}$ causes the linear terms in τ^{22} to be different from the expression for τ^{22} encountered in a linear analysis. This is due to the fact that the physical and tensoral (τ^{22}) components of the stress tensor are different. Their contribution in the equilibrium equations, however, will be the same.

Also, from the similarity in the form of τ^{33} and τ^{11}, the following is immediately obtained:

$$\tau^{33} = 2(\Phi + \Psi) w_{z'} + P' - 2(\Phi + \Psi) A + E \tag{E-44}$$

where

$$E = (\frac{u}{r})^2 + 2 \frac{u}{r} P' + (P'-\Phi-2\Psi)(\frac{u}{r})^2 + 2u_r(P'-\Phi-2\Psi)[2 \frac{u}{r} + (\frac{u}{r})^2]$$
$$+ (u_r^2 + w_r^2)[\Psi + (P'-\Phi-2\Psi)(1 + \frac{u}{r})^2]$$

and the first of the equilibrium equations, (E-34a and b), yields

$$2(\Phi+\Psi) \frac{\partial^2 u}{\partial r^2} + \frac{\partial P'}{\partial r} - 2(\Phi+\Psi) \frac{\partial A}{\partial r} + \frac{\partial B}{\partial r} + (\Phi+\Psi)(\frac{\partial^2 u}{\partial z'^2} + \frac{\partial^2 w}{\partial z' \partial r}) + \frac{\partial C}{\partial z'}$$
$$+ \frac{2}{r} (\Phi+\Psi) \frac{\partial u}{\partial r} + \frac{P'}{r} + [\frac{\partial u}{\partial r} - \frac{u}{r}][\frac{1}{r (1 + \frac{u}{r})}][2(\Phi+\Psi) \frac{\partial u}{\partial r} + P']$$
$$+ [1 + \frac{\partial u}{\partial r}][\frac{1}{r (1 + \frac{u}{r})}][-2 (\Phi+\Psi)A + B] -1/r [2(\Phi+\Psi)\frac{u}{r} + P']$$

$$- \frac{1}{r^2} \{[1 + \frac{\partial w}{\partial z'})^2 + (\frac{\partial u}{\partial z'})^2][\frac{\partial u}{\partial r} (1 + \frac{u}{r}) + \frac{u}{r}] + [2 \frac{\partial w}{\partial z'} + (\frac{\partial w}{\partial z'})^2$$

$$+ (\frac{\partial u}{\partial f'})^2][(1 + \frac{u}{r})(1 + \frac{\partial u}{\partial r})]\} \{2(\Phi+\Psi) \frac{u}{r} + P' + 2\Psi A + D\}$$

$$- \frac{1}{r} [2\Psi A+D] + \frac{1}{r^2 (1 + \frac{u}{r})^2} g^{13} g_{22,3} [2(\Phi+\Psi) \frac{u}{r} + P' + 2\Psi A + D]$$

+ [other nonlinear terms from expression E-34a].

This equilibrium equation may be written

$$2(\Phi+\Psi) \frac{\partial^2 u}{\partial r^2} + \frac{\partial P'}{\partial r} + (\Phi+\Psi) (\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 w}{\partial z' \partial r}) + \frac{2}{r} (\Phi+\Psi)(\frac{\partial u}{\partial r} - \frac{u}{r})$$

$$-2(\Phi+\Psi) \frac{\partial A}{\partial r} + \frac{\partial B}{\partial r} + \frac{\partial C}{\partial z'} + [\frac{\partial u}{\partial r} - \frac{u}{r}][\frac{1}{r(1 + \frac{u}{r})}][2(\Phi+\Psi) \frac{\partial u}{\partial r} + P']$$

$$-[1 + \frac{\partial u}{\partial r}][\frac{1}{r(1 + \frac{u}{r})}][2(\Phi+\Psi) A - B] - \frac{1}{r^2} \{[(1 + \frac{\partial w}{\partial z'}) + (\frac{\partial u}{\partial z'})^2]$$

$$[\frac{\partial u}{\partial r}(1+ \frac{u}{r}) + \frac{u}{r}] + [2 \frac{\partial w}{\partial z'} + (\frac{\partial w}{\partial z'})^2 + (\frac{\partial u}{\partial z'})^2][(1 + \frac{u}{r})(1 + \frac{\partial u}{\partial r})]\}$$

$$\{2(\Phi+\Psi) \frac{u}{r} + P' + 2\Psi A + D\} - \frac{1}{r} [2\Psi A + D] + \frac{1}{r^2(1 + \frac{u}{r})^2} g^{13} g_{22,3}$$

$$[2(\Phi+\Psi) \frac{u}{r} + P' + 2\Psi A + D] + 2\Gamma^1_{11}\tau^{11} + 3\Gamma^1_{31}\tau^{31} + \Gamma^2_{32}\tau^{31} + \Gamma^3_{33}\tau^{31}$$

$$+ \Gamma^3_{13} \tau^{11} + \Gamma^1_{33} \tau^{33} = 0 ,$$

(E-45)

and the second of the equilibrium equations, (E-37b), yields

$$2(\Phi + \Psi) \frac{\partial^2 w}{\partial z'} + \frac{\partial P'}{\partial z'} - 2(\Phi + \Psi) \frac{\partial A}{\partial z'} + \frac{\partial E}{\partial z'} + (\Phi + \Psi)(\frac{\partial^2 u}{\partial z'^2} + \frac{\partial^2 w}{\partial z' \partial r})$$

$$+ \frac{\partial C}{\partial z'} + \frac{1}{r} (\Phi+\Psi)(\frac{\partial u}{\partial z'} + \frac{\partial w}{\partial r}) + \frac{C}{r} + (\frac{\partial u}{\partial r} - \frac{u}{r})[\frac{1}{r(1 + \frac{u}{r})}]$$

$$[(\Phi+\Psi)(\frac{\partial u}{\partial z'} + \frac{\partial w}{\partial r}) + C] + 2\Gamma^3_{33}\ \tau^{33} + 3\Gamma^3_{13}\ \tau^{13} + \Gamma^2_{32}\ \tau^{33} + \Gamma^1_{11}\ \tau^{13}$$

$$+ \Gamma^1_{13}\ \tau^{33} + \Gamma^3_{11}\ \tau^{11} + \Gamma^3_{22}\ \tau^{22} = 0. \qquad \text{(E-46)}$$

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The purpose of this study was the establishment of analytical design procedures for laminated elastomeric bearings. This was approached with the application of the linear mathematical theory of elasticity and later with nonlinear large-deformation elasticity theory.

The linear theory yielded analytical approximations that are close to exact solutions and which are easily applied and evaluated. This analysis of one typical lamination yields the distribution of stress and deformation in the elastomer between "rigid" metal lamina. However, the limits of the linear elasticity theory are exceeded for greater than small bearing loads, indicating the need for the application of the more comprehensive large-deformation elasticity theory.

The large-deformation theory was stated and the equilibrium equations were derived, but the solution of these equations was not carried out.

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